Constraining conformal field theories with a higher spin symmetry

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We study the constraints imposed by the existence of a single higher spin conserved current on a three dimensional conformal field theory. A single higher spin conserved current implies the existence of an infinite number of higher spin conserved currents. The correlation functions of the stress tensor and the conserved currents are then shown to be equal to those of a free field theory. Namely a theory of N free bosons or free fermions. This is an extension of the Coleman-Mandula theorem to CFT's, which do not have a conventional S matrix. We also briefly discuss the case where the higher spin symmetries are "slightly" broken.

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1. Introduction

The classic Coleman-Mandula result [1], and its supersymmetric extension [2], states that the maximum spacetime symmetry of a theory with an S-matrix is the super-Poincare group¹. Interacting conformal field theories are interesting theories that do not have an S-matrix obeying the assumptions of [1,2]. In this paper we would like to address the question of whether a CFT can have a spacetime symmetry beyond the conformal group. We show that if a CFT has a conserved higher spin current, s > 2, then the theory is essentially free. Namely, all the correlators of the conserved currents are those of a free theory. In particular, this implies that the energy correlation function observables [3] of "conformal collider physics" [4] are the same as those of a free theory.

Let us clearly state the assumptions and the conclusions.

Assumptions:

- a) The theory is conformal and it obeys all the usual CFT axioms/properties, such as the operator product expansion, existence of a stress tensor, cluster decomposition, a finite number of primaries with dimensions less than some number, etc.
- a') The two point function of the stress tensor is finite.
- b) The theory is unitary.
- c) The theory contains a conserved current j_s of spin higher than two s > 2.
- d) We are in three spacetime dimensions.
- e) The theory contains unique conserved current of spin two which is the stress tensor.

The theorem, or conclusion:

There is an infinite number of even spin conserved currents that appear in the operator product expansion of two stress tensor. All correlation function of these currents have two possible structures. One is identical to that obtained in a theory of N free bosons, with currents built as O(N) invariant bilinears of the free bosons. The other is identical to those of a theory of N free fermions, again with currents given by O(N) invariant bilinears in the fermions.

Let us discuss the assumptions and conclusions in more detail.

We spelled out a') explicitly, to rule out theories with an infinite number of degrees of freedom, as in the $N = \infty$ limit of O(N) vector models, for example. Unitarity is a very important assumptions since it allows us to put bounds on the dimensions of operators,

¹ [2] also mentions the conformal group in the case of massless particles. However, [2] assumed that these massless particles are free in the IR, so that an S-matrix exists.

etc. We assumed the existence of a higher spin current. One might wonder if one could have a symmetry which is not generated by a local current. Presumably, a continuous symmetry implies the existence of an associated current, but, to our knowledge, this has not been proven². Assumption d) should hopefully be replaced by $d \geq 3$. Some of the methods in this paper have a simple extension to higher dimensions, and it should be straightforward to extend the arguments to all dimensions $d \geq 3$. In two dimensions, d = 2, we expect a richer structure. In fact, the Coleman-Mandula theorem in two dimensions allows integrable theories [5]. Also there are interesting current algebras in two dimensions which contain higher spin primaries beyond the stress tensor. An example are the W_N symmetry algebras [6]. The assumption of a unique stress tensor can also be relaxed, at the expense of making the conclusions a bit more complicated to state. It is really a technical assumption that simplifies the analysis. In fact, we actually generalize the discussion to the case that we have exactly two spin two conserved currents. In that case we have a kind of factorization into two subsectors and one, or both, could have higher spin currents. We expect something similar for a larger number of spin two currents, but we did not prove it. A simple theory when there are two spin two conserved currents is the product of a free theory with a non-trivial interacting theory. In this case all the higher spin currents live in the free subsector.

Now, regarding the conclusions, note that we did not prove the existence of a free field operator ϕ , in the CFT. The reason why we could not do it is easily appreciated by considering a theory of N scalar fields where we restrict the operators to be O(N) singlets. This is sometimes called "the free O(N) model". This theory obeys all the assumptions of our theorem (as well as the conclusions!) but it does not have a free field in the spectrum. Short of establishing the existence of free fields, we will show that the theory contains "bilocal" operators $B(x_1, x_2)$ and $F_-(x_1, x_2)$ whose correlators are the same as those of the free field operators $\sum_{i=1}^{N} : \phi^i(x_1)\phi^i(x_2) :$ and $\sum_{i=1}^{N} : \psi^i_-(x_1)\psi^i_-(x_2) :$ in a theory of free fields. Our statements also concern the infinite number of even spin conserved currents that appear in the operator product expansion of two stress tensors. We are not making any statement regarding other possible odd spin conserved currents, or spin two currents

Of course, if we have a theory given in terms of a Lagrangian, then Noether's theorem implies the existence of a current. Also if we assume we can generalize the action of the symmetry to the case that the parameter has some spacetime dependence, $\epsilon(x)$, then the usual argument implies the existence of an associated current, $j_{\mu} = \frac{\delta}{\delta \partial_{\mu} \epsilon}$, where the derivative is acting on the partition function, or the generating functional of correlation functions.

that do not appear the operator product of two stress tensors. Such currents, if present, are probably also highly constrained but we leave that to the future.

Notice that we start from the assumption of certain symmetries. By imposing the charge conservation identities we obtained the explicit form of the correlators. Thus, we can view this as a simple example of a realization of the bootstrap program. Namely, we never used "the Lagrangian", we derived everything from physical correlation functions and physical consistency conditions. Of course, the result turned out to be rather trivial since we get free theory correlation functions. We can also view this as an exercise in current algebra, now for currents of higher spin.

Recently, there have been some studies regarding the duality between various O(N) models in three dimensions and Vasiliev-type theories [7,8,9] in AdS_4 [10,11,12] (and dS_4 [13]). If one considers a Vasiliev theory with boundary conditions (at the AdS_4 boundary) which preserve the higher spin symmetry, then the theory obeys the assumptions of our theorem. Thus, our theorem implies that the theory is equivalent to a free theory of scalars or fermions. This was conjectured in [10] (see references therein for previous work), and tested in [11,12,14], see also [15,16]. In this context the quantization of N implies that the coupling constant of a unitary Vasiliev's theory is quantized, if we preserve the higher spin symmetry at the AdS boundary.

It is also interesting to consider the same Vasiliev theory but with boundary conditions that do not preserve the higher spin charge. An example was proposed in [10] by imposing a boundary condition for the scalar that produced the "interacting" O(N) theory, see also [14]. In such O(N) models, the higher spin currents acquire an anomalous dimension of order 1/N. Then the conclusions of our theorem do not apply. However, there are still interesting constraints on the correlators and we discuss some of them, as an example. It is likely that this method would give a way to compute correlators in this theory, but we will leave that for the future.

There is a large literature on higher spin symmetry, and we refer the reader to the review [17], which is also available in the arXiv [18,19,20,21,22,23].

1.1. Organization of the paper

In section two we discuss some generalities about higher spin currents.

In order to make the discussion clear, we will first present an argument that rules out theories with operators with low anomalous dimensions. This serves as a simple warm up example to the arguments presented later in the paper for the more general case. This is done in section three.

In section four we discuss some general properties of three point functions of currents, which will be necessary later.

We then present two slightly different approaches for showing the main conclusion. The first, in section five, requires a slightly more elaborate construction but it is computationally more straightforward. The other, in section six, is conceptually more straightforward, but it required us to use Mathematica a lot. We have presented both methods, since they could be useful for other purposes (useful spinoffs). These two sections can be read almost independently, and the reader should feel free to choose which one to read or skip first.

In section seven we discuss the case of slightly broken higher spin symmetry. We just discuss a couple of simple points, leaving a more general analysis to the future.

In section eight we discuss the case of exactly two spin two conserved currents.

In section nine we present some conclusions and discussion.

We also included several appendices with some more technical results, which could also be useful for other purposes.

2. Generalities about higher spin currents

We consider higher spin local operators $J_{\mu_1\cdots\mu_s}$ that transform in the representation of spin s of the rotation group. More specifically, if we consider the operator inserted at the origin, then it transforms in the spin s representation. It is convenient to define the twist, $\tau = \Delta - s$, where Δ is the scaling dimension of the operator. In three dimensions, the unitarity bound restricts the twist of a primary operator to [24]

$$\tau \ge \frac{1}{2},$$
 for $s = 0, \frac{1}{2}$

$$\tau \ge 1,$$
 for $s \ge 1.$ (2.1)

The equality in the first line corresponds to free fields: bosons and fermions. While in the second line it corresponds to conserved currents $\partial_{\mu}J^{\mu}{}_{\mu_{2}\cdots\mu_{s}}=0$.

From these conserved currents we can build conserved charges by contracting the current of spin s with an s-1 index conformal Killing tensor $\zeta^{\mu_1\cdots\mu_{s-1}}$, which obeys $\partial_{(\mu}\zeta_{\mu_1\cdots\mu_{s-1})}=0$, where the parenthesis denote the symmetric and traceless part. This

condition ensures that $\hat{J}_{\mu} = J_{\mu\mu_1\cdots\mu_{s-1}}\zeta^{\mu_1\cdots\mu_{s-1}}$ is a conserved current so that the associated charge $Q = \int_{\Sigma} *\hat{J}$ is conserved. A simple way to construct a conformal Killing tensor is to take a product of conformal Killing vectors [25]. In a CFT these charges annihilate the conformal invariant vacuum. These charges are conserved in the sense that their value does not depend on the hypersurface where we integrate the current. However, they might not commute with the Hamiltonian, since the Killing vectors can depend explicitly on the coordinates. This is a familiar phenomenon and it happens already with the ordinary conformal generators. These conserved currents lead to identities when they are inserted in correlation functions of other operators. Namely, imagine we consider an n point function of operators \mathcal{O}_i . Then we will get a "charge conservation identity" of the form

$$0 = \sum_{i} \langle \mathcal{O}_1(x_1) \cdots [Q, \mathcal{O}_i(x_i)] \cdots \mathcal{O}_n(x_n) \rangle$$
 (2.2)

If we know the action of Q on each operator this leads to certain equations which we call "charge conservation identities".

In this paper, we will focus exclusively on only one very particular charge that arises from a Killing tensor with only minus components. Namely, only $\zeta^{-\cdots}$ is nonzero, and the rest of the components are zero. Here x^- is a light cone coordinate. We think of the three coordinates as³ $ds^2 = dx^+ dx^- + dy^2$. We denote the charge associated to the spin s current by

$$Q_s = \int_{x^+ = \text{const}} dx^- dy \ j_s , \qquad j_s \equiv J_{s,-\dots}$$
 (2.3)

We have also defined j_s to be the current with all minus indices. Also from now on we will denote by $\partial \equiv \partial_{x^-}$.

From now on we will consider the twist operator τ defined so that it is the anomalous dimension minus the spin in the \pm directions (or boost generator in the \pm direction). Note that Q_s has spin s-1. One can check that it has twist zero. This is a very important property of this particular Q_s charge which we will heavily use.

One more general property that we will need is the following. The fact that Q_s has a non-trivial dimension (s-1) under the dilatation operator implies that

$$[Q_s, j_2] \propto \partial j_s + \cdots, \qquad s > 1$$
 (2.4)

 $^{^3}$ Throughout the paper we consider x^\pm as independent variables.

(recall that $\partial = \partial_{-}$). This is indicating that the current j_s , from which we formed the charge Q_s is appearing in the right hand side of the commutator (2.4). Here j_2 is the stress tensor of the theory. This is shown in more detail in appendix A. This fact is also related to the following. For any operator \mathcal{O} , the three point function $\langle \mathcal{O}\mathcal{O}j_2\rangle$ is nonzero, where j_2 is the stress tensor. This should be non-zero because the stress tensor generates conformal transformations. In particular, the stress tensor should always be present in the OPE of two identical operators. Here we are using that the two point function of the stress tensor is finite. (The stress tensor comes with a natural normalization so that it makes sense to talk about the coefficient in its two point function).

Note that two point functions of conserved currents are proportional to

$$\langle j_s(x_1)j_s(x_2)\rangle \propto \frac{(x_{12}^+)^{2s}}{|x_{12}|^{4s+2}}$$
 (2.5)

(Recall that our definition of j_s (2.3) contains only the minus components.)

3. Removing operators in the twist gap, $\frac{1}{2} < \tau < 1$

Notice that all operators with spin $s \ge 1$ should have twist bigger or equal to one. However, operators with spin $s = 0, \frac{1}{2}$ can have twists less than one. If $\tau = 1/2$, we have a free field which can be factored out of the theory. In this section we show that it is very easy to eliminate operators with twists in the range $\frac{1}{2} < \tau < 1$. We call this range the "twist gap". This will serve as a warm up exercise for the rest of the paper.

To keep the discussion simple, let us assume that we have a conserved current of spin four, j_4 . (We will show in the next section that a current of spin four is always present as soon as we have any higher spin current). Let us see how this current could act on a scalar operator of spin zero ϕ and twist in the twist gap, $\frac{1}{2} < \tau < 1$. The action of the charge Q_4 , $[Q_4, \phi]$, preserves the twist. Q_4 cannot annihilate ϕ since the operator product expansion of $\phi\phi$ contains the stress tensor (the unique spin two conserved current), and Q_4 cannot annihilate the stress tensor due to (2.4). The right hand side of $[Q_4, \phi]$ should be a combination of local operators and derivatives. We cannot have explicit functions of the coordinates since that would imply that Q_4 does not commute with translations. The fact that Q_4 commutes with translations can be seen from its integral expression (2.3) and the conservation of the current. The operator that appears in the right hand side has to have the same twist as ϕ . Thus, it should be a scalar. The only derivative that does not

change the twist is $\partial = \partial_{-}$. This derivative should be present three times due to spin conservation. In conclusion, we have that

$$[Q_4, \phi] = \partial_-^3 \phi \tag{3.1}$$

If we had many scalar operators of the same twist we would get $[Q_4, \phi_a] = c_{ab}\partial^3\phi_b$, where c_{ab} is symmetric due to the charge conservation identities for Q_4 acting on $\langle \phi_a \phi_b \rangle \propto \delta_{ab}$. Thus, we can diagonalize c_{ab} and we return to (3.1).

As the next step we can write the charge conservation identity for the action of Q_4 on the four point functions of these fields. It is convenient to do it in momentum space. Then we have

$$\left[\sum_{i=1}^{4} (k_{i-})^3\right] \langle \phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\rangle = 0$$
(3.2)

Using momentum conservation, and writing $k_i = k_{i-}$ for the minus component of the momentum, we find

$$\sum_{i=1}^{4} k_i^3 = -3(k_1 + k_2)(k_1 + k_3)(k_2 + k_3) = 0.$$
 (3.3)

when it acts on the four point function. Thus, the four point function is proportional to $\delta(k_{i_1} + k_{i_2})\delta(k_{i_3} + k_{i_4})$ and other two terms obtained by permutations. Together with rotational invariance and conformal invariance, this means that the four point function is a sum of two point functions

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \langle \phi(x_1)\phi(x_2)\rangle\langle \phi(x_3)\phi(x_4)\rangle + \text{permutations}$$
 (3.4)

Now we will show that the only operators that can obey (3.4) are free fields (see [26]). For that purpose we perform the ordinary operator product expansion as $|x_{12}| \to 0$. The first term in (3.4) gives the contribution of the identity operator. The other two terms are analytic around $x_{12} = 0$. Thus, they correspond to operators with twists $\tau_{int} = 2\tau + n$, with $n \ge 0$, where τ is the twist of ϕ . If this twist is $\tau > 1/2$, then the stress tensor, which has twist one, would not appear in this operator product expansion. But the stress tensor should always appear in the OPE of two identical currents. Thus, the only possible value is $\tau = 1/2$ and we have a free field. Such a free field will decouple from the rest of the theory.

Actually, by a similar argument we can eliminate spin 1/2 operators in the twist gap, $\frac{1}{2} < \tau < 1$. We repeat the above arguments for the particular component $\psi = \psi_{-}$ (the

spin 1/2 component as opposed to the spin -1/2). All the arguments go through with no change up to (3.3). That equation restricts the x^- dependence only. However, conformal symmetry and rotational invariance allow us to compute the four point function for any component ψ_{α} , and we find an answer that factorizes. Which then implies again $\tau = 1/2$ and a free fermion.

Note also that the analysis leading to (3.1) also constrains how all higher spin charges act on free fields, $[Q_s, \phi] = \partial^{s-1}\phi$, and similarly for fermions (we can show it easily for the ψ_- component and then the Dirac equation ensures that it acts in the same way on both components ψ_{α}^{4}).

At this point we should mention a qualitative argument for the reason that we get free fields. If we form wavepackets for the fields ϕ centered around some momentum and somewhat localized in space, then the action of the charge Q_4 will displace them by an amount which is proportional to the square of their momenta. Thus, if the wavepackets were colliding in some region, then these displacements would make them miss. Here we use that $d \geq 3$.

The results of this section can be very simply extended to $d \ge 3$. There the twist gap exists in the region $\frac{d-2}{2} \le \tau < d-2$, with the lower bound corresponding to free fields.

Returning to d = 3, in what follows, we will mostly consider the twist one operators that correspond to conserved currents. The idea will be similar, we first constrain the action of the higher spin charges and then determine the correlation functions of the operators.

We will present two independent ways of doing this. The first involves the notion of light cone operator product expansions and it is a bit more direct. It is presented in section five. The second involves a more explicit analysis of three point functions and it is conceptually easier, but computationally more complicated (needs use of Mathematica). It is presented in section six. The reader can choose which one to read and/or skip. But first some further generalities.

3.1. Action of the charges on twist one fields

Before jumping to those sections, let us make some preliminary statements on the action of charges Q_s on twist one operators. Since Q_s has twist zero, the only things we can have in the right hand side are other twist one fields. These include other (or the same) twist one fields. Even if we had fields in the twist gap $(\frac{1}{2} < \tau < 1)$, they cannot appear in

⁴ Here we again use that $[Q_s, P_{\mu}] = 0$.

the right hand side of a twist one current transformation, due to twist conservation and the fact that derivatives can change the twist, at most by integer amounts. Thus, we have the general transformation law

$$[Q_s, j_{s'}] = \sum_{s''=\max[s'-s+1,0]}^{s'+s-1} \alpha_{s,s',s''} \partial^{s'+s-1-s''} j_{s''}$$
(3.5)

(the limits of the sum are explained below). Here we wrote explicitly arbitrary constants $\alpha_{s,s',s''}$ appearing in front of each term, below we will assume them implicitly. We will also sometimes omit the derivatives. The sum (3.5) can involve twist one and s = 0, 1/2 fields, which are not conserved currents, but we still denote them by j_0 , $j_{1/2}$. Here everything has minus indices which we have not indicated. The derivatives are also minus derivatives. The number of derivatives is easily fixed by matching the spin on both sides.

One easy property we can prove is the following, if current j_y appears in the right hand side of $[Q_s, j_x]$, then we should have that current j_x appears in the right hand side of $[Q_s, j_y]$. This follows by considering the action of Q_s on the $\langle j_x j_y \rangle$ two point function and (2.5). In summary,

$$[Q_s, j_x] = \partial^{s+x-y-1} j_y + \dots \quad \Rightarrow \quad [Q_s, j_y] = \partial^{s+y-x-1} j_x + \dots \tag{3.6}$$

This has a simple consequence which is the fact that the spread of spins s'' in (3.5) is as indicated in (3.5). The upper limit in (3.5) is obvious. The lower limit in (3.5) arises from the fact that the current $j_{s'}$ should appear in the right hand side of $[Q_s, j_{s''}]$, and the upper limit in this commutator, results in the lower limit in (3.5).

4. Basic facts about three point functions

For our analysis it is important to understand the structure of the three point functions of conserved currents and other operators in d = 3.

This problem was analyzed in [27] (see also [28]). Unfortunately, it is not clear to us what was proven and what was the result of a case by case analysis with a later extrapolation (a "physics" proof). Thus, in appendix I we prove the results that will be crucial for section five. These only involve certain light-like limits of the correlators. For section six we need some particular cases with low spin, where the statements in [27,28] can be explicitly checked.

Nevertheless, let us summarize the results of [27]. The three point function of the conserved currents has, generically, three distinct pieces

$$\langle j_{s_1}j_{s_2}j_{s_3}\rangle = \langle j_{s_1}j_{s_2}j_{s_3}\rangle_{boson} + \langle j_{s_1}j_{s_2}j_{s_3}\rangle_{fermion} + \langle j_{s_1}j_{s_2}j_{s_3}\rangle_{odd}. \tag{4.1}$$

Here the piece $\langle j_{s_1}j_{s_2}j_{s_3}\rangle_{boson}$ is generated by the theory of free bosons. The piece $\langle j_{s_1}j_{s_2}j_{s_3}\rangle_{fermion}$ is generated by the theory of free fermion and the piece $\langle j_{s_1}j_{s_2}j_{s_3}\rangle_{odd}$ is not generated by free theories. The boson and fermion pieces are known in closed form while the odd piece can be computed in each particular case explicitly by imposing the conservation of the currents. By considering particular examples it was observed that the odd piece is non-zero whenever the triangle rule is satisfied

$$s_i \le s_{i+1} + s_{i+2}. \tag{4.2}$$

In appendix B we derive an integral expression for these odd correlators that naturally incorporates this triangle rule. We start with free higher spin conformal fields in four dimensions, introduce a conformal invariant perturbation, and look at correlators on a three dimensional slice. The correlators automatically obey the conservation condition and are conformal invariant. See appendix B for more details.

Another property of three point functions which we found useful is that any three point function of the two identical⁵ currents with the third one that has an odd spin is zero

$$\langle j_s j_s j_{s'} \rangle = 0$$
 for s' odd (4.3)

This is easy to check for the explicit expressions for the boson and fermion solutions. It is also can be seen from the integral representation for the odd piece.

4.1. General expression for three point functions

The space of three point functions of conserved currents is conveniently split into two parts according to their transformation under the parity. Practically, we call "even" ones the three point functions that do not contain the three dimensional epsilon tensor and "odd" the ones that do.

The parity even correlation functions are generated by

$$\mathcal{F}_{even} = e^{\frac{1}{2}(Q_1 + Q_2 + Q_3)} e^{P_1 + P_2} (b \cosh P_3 + f \sinh P_3)$$
(4.4)

 $^{^{5}}$ We need literally the same current. It is not enough that they have the same spin.

where the piece proportional to b stands for correlation functions in the theory of free bosons and the one proportional to f for the ones in the free fermion theory⁶. The P_i and Q_i are some cross ratios whose form can be found in [27]. For currents which have indices only along the minus direction we can find the simple expressions

$$Q_i = \ell_i^2 (\hat{x}_{i,i+1}^+ - \hat{x}_{i,i-1}^+) , \quad P_i = \ell_{i+1} \ell_{i-1} \hat{x}_{i-1,i+1}^+ , \quad \hat{x}^+ = \frac{x^+}{x^2}$$
 (4.5)

where ℓ_i are the parameters in the generating function. In other words the term with $\ell_1^{2s_1}$ gives us the current $j_{s_1}(x_1)$, etc.

When spins of the currents are integer (4.4) reproduces the results given in [27], but (4.4) also gives the answer for half integer spin currents⁷.

Due to the different behavior under $P_3 \to -P_3$ the bosonic and fermionic parts never mix inside the charge conservation identities for three point functions that we will consider. Thus, whenever we can solve the charge conservation identities they are satisfied separately for f = 0 and b = 0 parts.

5. Argument using bilocal operators

Here we present an argument for the main conclusion.

5.1. Light cone limits of correlators of conserved currents

We consider the light cone OPE limit of two conserved currents $j_s(x)j_{s'}(0)$. All indices are minus. We will consider the twist one contribution to the OPE⁸. This can be cleanly separated from the lower twist contribution which can only arise from the identity or spin

$$\mathcal{F}_{even} = e^{\frac{1}{2}(Q_1 + Q_2 + Q_3)} e^{(P_1 + P_2 + P_3)} \times \left(b + f + \frac{b - f}{3} [e^{-2P_1} + e^{-2P_2} + e^{-2P_3}] \right)$$

 $^{^{6}}$ An equivalent, but symmetric version of (4.4) is given by

⁷ Note that for half integer spins the labels b and f just give two different possible structures, since, of course we need both bosons and fermions to have half integer spins

⁸ Here we consider the contribution of twist *exactly* one. This should not be confused with discussions in weakly coupled theories where one is interested in the whole tower of operators which have twists close to one (or in four dimensions twist close to two). Such operators appear in discussions of deep inelastic scattering and parton distribution functions.

 $s=0,\frac{1}{2}$ operators. This clean separation is possible because there is a finite number of operators in the twist gap, $\frac{1}{2} < \tau < 1$. To extract the twist one contribution of a given spin, we can consider the three point functions $\langle j_s(x_1)j_{s'}(x_2)j_{s''}(x_3)\rangle$. We can also consider the contribution of all spins together by taking the following light-like limits.

We take the limit $x_{12}^+ \to 0$ first. Then we take the limit $y_{12} \to 0$. By twist conservation, the correlator could behave only like $1/|y_{12}|$ or like $1/y_{12}$, in this limit. The fermion piece vanishes when we take $x_{12}^+ \to 0$. One can see that the $1/y_{12}$ behavior can be produced only by a parity odd piece. This follows from the fact that we can put the third current at $y_3 = 0$, but at generic x_3^{\pm} . This still allows us to extract any possible twist one current that appears in the OPE limit. However, the parity transformation $y \to -y$ would change the sign of the correlator only if we have a $1/y_{12}$ term in the OPE. Below, the part that contains the $1/|y_{12}|$ behavior is called the "boson" piece. It comes from the boson piece of the three point functions.

Similarly, we can look at the piece that goes like x_{12}^+ in the limit that x_{12}^+ goes to zero. Then we extract the piece going like $\frac{x_{12}^+}{|y_{12}|^3}$.

More explicitly, these limits are defined by

$$\underline{j_s j_{s'}}_b = \left(\lim_{y_{12} \to 0^+} + \lim_{y_{12} \to 0^-}\right) |y_{12}| \lim_{x_{12}^+ \to 0} j_s(x_1) j_{s'}(x_2) - (\text{lower})$$

$$\underline{j_s j_{s'}}_f = \left(\lim_{y_{12} \to 0^+} + \lim_{y_{12} \to 0^-}\right) \lim_{x_{12}^+ \to 0} \frac{|y_{12}|^3}{x_{12}^+} \left[j_s(x_1) j_{s'}(x_2) - \frac{1}{|y_{12}|} \underline{j_s j_{s'}}_b - (\text{lower})\right]$$
(5.1)

where we have indicated by an underline the fact that we take the light-like limit (and the subindex on the line reminds us which of the two limits we took). We will also indicate by an underline points that are light-like separated along x^- . In the fermion like limit (5.1), we have indicated that we extracted the boson piece. However, in practice we will only use it when the boson piece vanishes. Thus, there is no ambiguity in extracting the limit. We also extract any possible twist less than one operators that could appear. There is a finite list of them. Thus, there is no problem in extracting the lower twist operators. In section three we have eliminated these lower twist operators by making the assumption of the existence of Q_4 . Here we have not yet derived the existence of Q_4 , so we had to argue that these lower twist operators do not affect the definition of the limits and the extraction of the twist one contribution to the OPE. In fact, from now on, these lower twist contribution play no role. After we show the existence of Q_4 , we can then remove

them using section three. We could define also a limit that extracts the odd piece, but we will not need it.

The three point functions simplify dramatically in this limit. For the boson part we have

$$\left(\lim_{y_{12}\to 0^{+}} + \lim_{y_{12}\to 0^{-}}\right) \lim_{x_{12}^{+}\to 0} |x_{12}| \langle j_{s_{1}}j_{s_{2}}j_{s_{3}}\rangle_{b} \propto \partial_{1}^{s_{1}}\partial_{2}^{s_{2}} \langle \underline{\phi\phi^{*}}j_{s_{3}}\rangle_{free} ;$$

$$\langle \underline{\phi\phi^{*}}j_{s_{3}}\rangle_{free} \propto \frac{1}{\sqrt{\hat{x}_{13}\hat{x}_{23}}} \left(\frac{1}{\hat{x}_{13}} - \frac{1}{\hat{x}_{23}}\right)^{s_{3}} \tag{5.2}$$

where $\langle \underline{\phi} \phi^* j_{s_3} \rangle_{free}$ stands for the correlation function of a free complex boson. The underline reminds us that the points are light-like separated, but there is no limit involved. The reason we take a *complex* boson is to allow for non-zero values when s is odd. We have displayed only the x_i^- dependence⁹ and we have defined a slightly shifted version of the coordinates

$$\hat{x}_1 = x_1^-, \qquad \hat{x}_2 = x_2^-, \qquad \hat{x}_3 = x_3^- - \frac{y_{13}^2}{x_{13}^+}$$
 (5.3)

We should emphasize that in (5.2) the expression $\langle \phi \phi^* j_{s_3} \rangle_{free}$ denotes a correlator in a theory of a free complex boson. In particular, j_{s_3} is not the original current in the unknown theory, but the current of the free boson theory.

For a free boson theory, it is easy to see why (5.2) is true. For a free boson the currents are given by expressions like $\sum \partial^i \phi \partial^{s-i} \phi^*$. When we take the limit $x_{12}^+ \to 0$ only one term survives from the two sums associated to j_{s_1} and j_{s_2} (recall that $\partial = \partial_-$). The reason is that we need a contraction of the scalar with no derivatives in order *not* to bring down a factor x_{12}^+ in the numerator. That single term contains all the derivatives on the ϕ fields that are contracted with j_{s_3} .

Actually, without assuming the explicit forms given in [27], one can prove that (5.2) follows, in the light cone limit, from conformal symmetry and current conservation for the third current. Since this is a crucial point for our arguments, we provide an explicit proof of (5.2) in appendix I.

Using the same reasoning for the fermion part we obtain (see appendix I)

$$\left(\lim_{y_{12}\to 0^{+}} + \lim_{y_{12}\to 0^{-}}\right) \lim_{x_{12}^{+}\to 0} \frac{|x_{12}|^{3}}{x_{12}^{+}} \langle j_{s_{1}}j_{s_{2}}j_{s_{3}}\rangle_{f} \propto \partial_{1}^{s_{1}-1}\partial_{2}^{s_{2}-1} \langle \underline{\psi}\underline{\psi}^{*}j_{s_{3}}\rangle_{free} ;
\langle \underline{\psi}\underline{\psi}^{*}j_{s_{3}}\rangle_{free} \propto \frac{1}{(\hat{x}_{13}\hat{x}_{23})^{3/2}} \left(\frac{1}{\hat{x}_{13}} - \frac{1}{\hat{x}_{23}}\right)^{s_{3}-1}$$
(5.4)

⁹ There is also a factor of $\frac{1}{x_{13}^+} = \frac{1}{x_{23}^+}$ in the right hand side of (5.2).

where ψ stands for a free complex fermion. If s = 0, then $\langle \underline{\psi} \psi^* j_0 \rangle_{free} = 0$, and the limit in the first line of (5.4) vanishes. This is simply the statement that in a free fermion theory we do not have a twist one spin zero operator.

As an aside, for the odd piece one can obtain a similar expression, given in appendix B. This will not be necessary here, since the higher spin symmetry will eliminate the odd piece and we will not need to know its explicit form.

5.2. Getting an infinite number of currents

Imagine we have a current of spin s. From (2.4) we know that $[Q_s, j_2] = \partial j_s + \cdots$. Then (3.6) implies that

$$[Q_s, j_s] = \partial^{2s-3} j_2 + \cdots \tag{5.5}$$

Let us first

assume that
$$\langle j_2 j_2 j_2 \rangle|_b \neq 0$$
 (5.6)

We now consider the charge conservation identity for Q_s acting on $\langle \underline{j_2j_2}_b j_s \rangle$, where we mean a charge conservation identity that results from acting with Q_s on $\langle \underline{j_2j_2}_b \rangle$ and then taking the light cone limit for the first two variables. In other words, we get

$$0 = \langle [Q_s, j_2] j_2 j_s \rangle + \langle j_2 [Q_s, j_2] j_s \rangle + \langle j_2 j_2 [Q_s, j_s] \rangle$$

$$(5.7)$$

In the first two terms we first expand the commutator using (3.5) and then take the light cone limit using (5.2). We will consider first the case of integer s. Then the first two terms in (5.7), produce

$$\langle [Q_s, j_2] j_2 j_s \rangle + \langle j_2 [Q_s, j_2] j_s \rangle = \partial_1^2 \partial_2^2 \left[\gamma \partial_1^{s-1} + \delta \partial_2^{s-1} \right] \langle \underline{\phi \phi^*} j_s \rangle_{free}$$
 (5.8)

Here the symmetry under $x_1 \leftrightarrow x_2$ implies that $\gamma = (-1)^s \delta$, due to the symmetries of (5.2). In the third term in (5.7), when Q_s acts on j_s , we generate many possible operators through (3.5). Each of those currents might or might not have an overlap with $\underline{j_2 j_2}_b$. Thus, combining this with (5.8) we conclude that the charge conservation identity that results from acting with Q_s on $\langle \underline{j_2 j_2}_b j_s \rangle$ is

$$0 = \partial_1^2 \partial_2^2 \left[\gamma (\partial_1^{s-1} + (-1)^s \partial_2^{s-1}) \langle \underline{\phi \phi^*} j_s \rangle_{free} + \sum_{k=1}^{2s-1} \tilde{\alpha}_k \partial_3^{2s-1-k} \langle \underline{\phi \phi^*} j_k \rangle_{free} \right]$$
(5.9)

where $\tilde{\alpha}_k$ is the product of the constants in (3.5) and the actual value of the boson part of the three point function. The overall derivatives can be removed since the right hand side

is a sum of terms of the form $\frac{1}{x_{13}^a x_{23}^b}$ where a and b are half integer, so if a term was non-zero before the derivative acted it would be non-zero afterwards, and terms with different powers cannot cancel each other. Notice that (5.9) involves the explicit functions defined in (5.2). An important property of these functions is that $\langle \underline{\phi} \phi^* j_k \rangle_{free}$ has a zero of order k when $x_{12}^- \to 0$, and the ∂_3 derivative does not change the order of this zero. This implies that all the terms in the last sum in (5.9) are independent. Thus, all the $\tilde{\alpha}_k$ are fixed. We know that $\tilde{\alpha}_2$ is nonzero due to (5.5) and (5.6). Here we used that the spin two current is unique so that there is no other contribution that could cancel the j_2 contribution. Given that this term is non-zero, then γ is nonzero. The $\tilde{\alpha}_k$ for odd k are set to zero by the $x_1 \leftrightarrow x_2$ symmetry of the whole original equation, (5.7). The rest of the $\tilde{\alpha}_k$ are fixed, and are equal to what we would obtain in a free theory, which is the unique solution of (5.9). It is then possible to check that all $\tilde{\alpha}_k$, with $k = 2, 4, \dots, 2s - 2$ are non-zero. The explicit proof of this statement is given in appendix J.

In particular, this implies that the current j_4 is in the spectrum, and it is in the right hand side of the $\underline{j_2j_2}_b$ OPE. This is true regardless of whether the original s is even or odd. Now we can go back to section three and remove the operators in the twist gap, $\frac{1}{2} < \tau < 1$.

Since s > 2, we find that 2s - 2 > s, so that we are also finding currents with spins bigger than the original one. Thus, repeating the argument we find an infinite number of conserved currents.

We now show that a twist one scalar j_0 is present in the spectrum, and in the right hand side of $\underline{j_2j_2}_b$. This can be shown by considering the Q_4 charge conservation identity acting on $\langle \underline{j_2j_2}_b j_2 \rangle$. This leads to an expression very similar to (5.9), but with different limits in the sum (see (3.5)), such that now a non-zero $\tilde{\alpha}_4$ implies the existence of j_0 . Now, considering the ward identity from Q_4 on $\langle \underline{j_2j_2}_b j_0 \rangle$ we show that j_2 is in the right hand side of

$$[Q_4, j_0] = \partial^3 j_0 + \partial j_2 + \cdots \tag{5.10}$$

where the dots denote other twist one operators that have no overlap with $\underline{j_2j_2}_b$ which could possibly appear. Let us now show that the fermion components are zero. We consider the charge conservation identity from Q_4 on $\langle \underline{j_2j_2}_f j_0 \rangle$. Notice that $\langle \underline{j_2j_2}_f j_0 \rangle = 0$, see (5.4). Thus, this charge conservation identity implies that $\langle \underline{j_2j_2}_j z_2 \rangle_f = 0$. Here we have used that there is a unique stress tensor since we assumed that the j_2 appearing at various places is always the same. Thus, if $\langle \underline{j_2j_2j_2}\rangle_b \neq 0$, then $\langle \underline{j_2j_2j_2}\rangle_f = 0$. Conversely, if $\langle \underline{j_2j_2j_2}\rangle_f \neq 0$, then $\langle \underline{j_2j_2j_2}\rangle_b = 0$, and the discussion after (5.9) implies that also $\langle \underline{j_2j_2}_b j_s \rangle = 0$ for all s.

In the case that $\langle j_2 j_2 j_2 \rangle_f$ is nonzero, we can start with a ward identity similar to (5.9), but for $\langle j_2 j_2 f_j j_s \rangle$. We find an identical conclusion, an infinite set of even spin currents in the right hand side of $j_2 j_2 f_j$. (See appendix J for an explicit demonstration). In conclusion, due to the uniqueness of the stress tensor we have only one of two cases:

$$\langle j_2 j_2 j_2 \rangle_b \neq 0 , \Rightarrow \langle j_2 j_2 j_2 \rangle_f = 0 , \quad \underline{j_2 j_2}_b = \sum_{k=0}^{\infty} [j_{2k}] , \quad \underline{j_2 j_2}_f = 0$$

$$\langle j_2 j_2 j_2 \rangle_f \neq 0 , \Rightarrow \langle j_2 j_2 j_2 \rangle_b = 0 , \quad \underline{j_2 j_2}_f = \sum_{k=1}^{\infty} [j_{2k}] , \quad \underline{j_2 j_2}_b = 0$$

$$(5.11)$$

where the brackets denote currents and their derivatives. In other words, the brackets are the contribution of the conformal block of the current j_s to the twist one part of the OPE.

It is also possible to start with a half integer s, higher spin conserved current j_s and to obtain a charge conservation identity similar to (5.9) (we describe it in detail in appendix D). This again shows that currents with even spins $k = 2, 4, \dots 2s - 1$ appear in the right hand side of $\underline{j_2j_2}_b$ or $\underline{j_2j_2}_f$. Again, if $s \geq 5/2$ we get higher spin conserved currents with even spins and we return to the previous case.¹⁰

In the next two subsections we consider the first case in (5.11) and then the second.

5.3. Definition of bilocal operators

In the light-like limit we encountered a correlator which was essentially given by a product of free fields (5.2). Here we will argue that we can "integrate" the derivatives and define a bilocal operator $B(x_1, x_2)$ with x_1 and x_2 separated only along the minus direction.

Now, let us make a comment on light cone OPE's. The light cone OPE of two currents $\underline{j_s j_{s'}}_b$, contains only twist one fields of various spins. The overlap with each individual current is computed by (5.2). In particular, all such limits will contain a common factor of $\partial_1^s \partial_2^{s'}$. We can "integrate" such derivatives and define a quasi-bilocal operator $\hat{B}(\underline{x_1, x_2})$ such that

$$\underline{j_s j_{s'_b}} = \partial_1^s \partial_2^{s'} \hat{B}(\underline{x_1, x_2}) \tag{5.12}$$

¹⁰ If we have a half integer higher spin current, we will also have a supercurrent, a spin 3/2 current. This then requires that both $\langle j_2 j_s j_2 \rangle_b$ and $\langle j_2 j_2 j_2 \rangle_f$ are nonzero. Since we had the dichotomy (5.11), we should have more than one spin two conserved current if there is any half integer higher spin current.

The underline reminds us that x_1 and x_2 are null separated (along the x^- direction). Here the right hand side is simply a superposition of twist one fields and their derivatives. They are defined such that

$$\langle j_s j_{s'_h} j_{s''}(x_3) \rangle = \partial_1^s \partial_2^{s'} \langle \hat{B}(x_1, x_2) j_{s''}(x_3) \rangle$$
 (5.13)

Of course, we also see that $\langle \hat{B}(\underline{x_1, x_2}) j_s(x_3) \rangle \propto \langle \underline{\phi(x_1)} \phi^*(x_2) j_{s_3}(x_3) \rangle_{free}$. This implies that \hat{B} transforms as two weight 1/2 fields under conformal transformations.

A particular one we will focus on is the quasi-bilocal that we get from the stress tensor, defined by

$$\underline{j_2 j_{2h}} = \partial_1^2 \partial_2^2 B(\underline{x_1, x_2}) \tag{5.14}$$

These are reminiscent of the (Fourier transform along x^- of the) operators whose matrix elements define parton distribution functions. One very important difference is that here we are constructing B only from the operators that have twist exactly equal to one.

We can define similar operators in the case of fermions

$$\frac{j_{s}j_{s'}}{j_{2}j_{2}} = \partial_{1}^{s-1}\partial_{2}^{s'-1}\hat{F}_{-}(\underline{x_{1}, x_{2}}),
\underline{j_{2}j_{2}}_{f} = \partial_{1}\partial_{2}F_{-}(\underline{x_{1}, x_{2}})$$
(5.15)

Here F_{-} transforms as the product of two free fermions with minus polarizations: $\psi_{-}(x_1)\psi_{-}(x_2)$:. It is defined again through (5.4), by "integrating" the derivatives. It is the superposition of all the (even spin) currents that appear in the right hand side of $\underline{j_2j_2}_f$.

We call these operators "quasi-bilocals" because they transform under conformal transformations as a product of two elementary fields. This does not mean that they are honest local operators, in the sense that their correlators have the properties of products of fields, with singularities only at the insertion of other operators. This is illustrated in appendix H. Our task will be to show that in a theory with higher spin symmetry, they become true bilocal operators, with correlators equal to the free field ones. The first step is to constrain the action of Q_s on B and F_- in (5.14), (5.15).

5.4. Constraining the action of the higher spin charges

Here we will show that

$$[Q_s, B(x_1, x_2)] = (\partial_1^{s-1} + \partial_2^{s-1})B(x_1, x_2)$$
(5.16)

Then we will show that this implies that their correlators have the free field form. Let us first assume that $\langle j_2 j_2 j_2 \rangle_b$ is nonzero. In (5.16) we consider charges Q_s with s even constructed out of the currents j_s that appear in the right hand side of $\underline{j_2 j_2}_b$. As we saw before, there is an infinite number of such currents.

We would like to compute $[Q_s, B(\underline{x_1, x_2})]$. We can compute $[Q_s, \underline{j_2, j_2}_b] = [Q, \underline{j_2}]\underline{j_2}_b + \underline{j_2[Q, \underline{j_2}]_b}$. In other words, the action of Q_s commutes with the limit. This follows from the charge conservation identity of Q_s on $\langle \underline{j_2}\underline{j_2}\underline{j_k}\rangle$ and taking the light cone limit, which gives

$$\langle [Q_s, j_2]j_2j_k\rangle + \langle j_2[Q_s, j_2]j_k\rangle = -\langle \underline{j_2j_2}[Q_s, j_k]\rangle = \langle [Q_s, \underline{j_2j_2}]j_k\rangle$$
 (5.17)

So, we can write $[Q, j_2]$ in terms of currents and derivatives (with indices and derivatives all along the minus directions). Thus, in the end we can use the formula (5.13) to write

$$[Q_s, B(x_1, x_2)] = (\partial_1^{s-1} + \partial_2^{s-1})\tilde{B}(x_1, x_2) + (\partial_1^{s-1} - \partial_2^{s-1})B'(x_1, x_2)$$
(5.18)

 \tilde{B} contains all the even currents and it is symmetric under the interchange of x_1 and x_2 and B' contains the odd currents and it is antisymmetric under the interchange. Of course, the full expression is symmetric under this interchange.

Let us first show that B' is zero. Since B' is odd, it includes only odd currents $j_{s'}$ with s' odd. Imagine that B' contains a particular odd current $j_{s'}$. Then, if s' > 1, we consider the charge conservation identity coming from the action of $Q_{s'}$ on $\langle B'j_2\rangle$

$$0 = \langle [Q_{s'}, B'] j_2 \rangle + \langle B'[Q_{s'}, j_2] \rangle$$

$$0 = \gamma (\partial^{s'-1} - \partial^{s'-1}) \langle \underline{\phi \phi^*} j_2 \rangle + \sum_{k=0}^{s'+1} \tilde{\alpha}_k \partial_3^{s'+1-k} \langle \underline{\phi \phi^*} j_k \rangle$$
(5.19)

This charge conservation identity is very similar, in structure, to the one we considered in (5.9). Namely, the last term contains a sum over various currents which give rise to different functional forms. Therefore all $\tilde{\alpha}_k$'s are fixed. $\tilde{\alpha}_{s'}$ is nonzero because $[Q_{s'}, j_2]$ contains $j_{s'}$, (2.4). The rest of the terms are fixed to the same coefficients that we would have if we were considering the same charge conservation identity in a free theory of a

complex boson¹¹. In particular, as shown in appendix J, $[Q_{s'}, j_2]$ produce a j_1 whose overlap with B' is nonzero. Once we have shown that j_1 appears in B', we can consider the charge conservation identity for Q_s on $\langle Bj_1 \rangle$.

$$0 = \langle [Q_s, Bj_1] \rangle = (\partial_1^{s-1} - \partial_2^{s-1}) \langle B'j_1 \rangle + \langle B[Q_s, j_1] \rangle$$

$$(5.20)$$

Here the first term is non-zero. Analyzing this charge conservation identity, one can show that it can be obeyed, for a non-zero first term, only if $[Q_s, j_1]$ contains a j_s in the right hand side. The analysis of this charge conservation identity is somewhat similar to the previous ones and it is discussed in more detail in appendix J. Here we have used that Q_s is a current that appears in the right hand side of $\underline{j_2j_2}_b$. In other words, Q_s is built out of the same current j_s that appears in the right hand side of $[Q_s, j_1]$. This implies that $\langle j_s j_s j_1 \rangle$ would need to be non-zero¹². However all such three point functions are zero when the two j_s currents are identical (4.3). Since we reached a contradiction, we conclude that B' is zero.

We would like now to show that \tilde{B} is the same as B. We know that \tilde{B} is non-zero because we can consider the charge conservation identity for Q_s on $\langle Bj_2 \rangle$, and use that j_s appears in the right hand side of $[Q_s, j_2]$. This implies that $\langle \tilde{B}j_2 \rangle$ is nonzero. In fact, we can normalize it in such a way that j_2 appears in the same way on B and \tilde{B} . Then we can consider $B - \tilde{B}$. This does not contain a j_2 . Here we are using that there is a unique j_2 in the theory. Say that $j_{s'}$, with s' even and s' > 2, is a candidate current to appear in the right hand side of $B - \tilde{B}$. We can consider the $Q_{s'}$ charge conservation identity for $\langle (B - \tilde{B})j_2 \rangle$. This charge conservation identity has a form similar to (5.19),

$$0 = \gamma (\partial_1^{s'-1} + \partial_2^{s'-1}) \langle \phi \phi^* j_2 \rangle + \sum_{k=0}^{s'+1} \tilde{\alpha}_k \partial_3^{s'+1-k} \langle \phi \phi^* j_k \rangle$$
 (5.21)

Here we are assuming that $\tilde{\alpha}_{s'}$ is non-zero. However, this equation would also show that $\tilde{\alpha}_2$ is non-zero which is now in contradiction with the fact that $B - \tilde{B}$ does not contain j_2 (see appendix J). So we have shown that $B - \tilde{B}$ cannot have any current of spin s' > 0.

In (5.19) all α_k with even k are set to zero.

Let's emphasize again we consider j_s that appears in the OPE of $j_2j_2 \sim j_s$. Then, in principle, we can have $[Q_s, j_1] = \alpha j_s + \beta \tilde{j_s} + \dots$ The charge conservation identity (5.20) implies that α is non-zero. This implies that $\langle j_s j_s j_1 \rangle$ must be non-zero.

Let us now focus on a possible spin zero operator j'_0 . We put a prime to distinguish it from j_0 which is the one that appears in B. j'_0 might or might not be equal to j_0 . If $\langle j_0 j'_0 \rangle \neq 0$, then we can consider the charge conservation identity for Q_4 acting on $\langle (B - \tilde{B})j_0 \rangle$. Using (5.10) we get a non-zero term when Q_4 acts on j_0 producing j_0 . However, this charge conservation identity cannot be obeyed if we are setting the term involving j_2 to zero. Thus, $\langle j_0 j'_0 \rangle = 0$. Then there is some even current in the $j_2 j_2$, call it s'', such that $[Q_s, j_{s''}] \sim j'_0 + \cdots$. From (3.5) this implies that s'' < s. Then we consider the action of Q_s on $\langle (B - \tilde{B})j_{s''} \rangle$. This action produces (up to derivatives) both $\langle (B - \tilde{B})j'_0 \rangle$ and $\langle (B - \tilde{B})j_2 \rangle$. But given that the second is zero, then the first is also zero. This charge conservation identity is very similar to the others we have been discussing (see appendix J). Here it is important that s'' < s, so that we get a constraint on $\langle (B - \tilde{B})j'_0 \rangle$. So we conclude that there cannot be any j'_0 in $B - \tilde{B}$.

In conclusion, we have shown that (5.16) holds when x_1 and x_2 are null separated along the minus direction.

Now, once we argued that (5.16) is true, then we can consider any n point function of bilinears $\langle B(\underline{x_1, x_2}) \cdots B(\underline{x_{2n-1}, x_{2n}}) \rangle$. We have an infinite number of constraints from all the conserved charges (5.16). These constraints take the form

$$\sum_{i=1}^{2n} \partial_i^{s-1} \langle B(\underline{x_1, x_2}) \cdots B(\underline{x_{2n-1}, x_{2n}}) \rangle = 0 , \qquad s = 2, 4, 6, \cdots$$
 (5.22)

where $\partial_i = \partial_{x_i^-}$. This constrains the x_i^- dependence of this correlator. We show in appendix E that this constrains the correlator to be a sum of functions of differences $x_i^- - x_j^-$, $x_l^- - x_k^-$. More explicitly, the x_i^- dependence is such that the correlator is a sum of functions of the form $\sum_{\sigma} g_{\sigma}(x_{\sigma(1)}^- - x_{\sigma(2)}^-, \cdots, x_{\sigma(2n-1)}^- - x_{\sigma(2n)}^-)$, where σ are the various permutations of 2n elements. In principle, these functions, g_{σ} are all different.

In addition, we know that the correlator should respect conformal symmetry and rotational invariance. Thus, it should be a function of conformal dimension 1/2 at each location and a function of distances d_{ik} . Of course, $d_{ij}^2 = x_{ij}^+ x_{ij}^- + (y_i - y_j)^2$. Thus, we can now write the function in terms of distances. Let us explain this point a bit more. We have defined the bilocals in terms of an operator product expansion. But we have also noticed that we can also view these bilocals as special conformal block like objects written in terms of currents. From that perspective, the transformation properties under conformal transformations are identical to those of a product of bosonic fields. This is what we are using at this point. Thus, for each function g_{σ} , say $g(x_{13}^-, x_{24}^-, \cdots)$ we can now

write $\hat{g}(d_{13}, d_{24}, \cdots)$. But in addition, under conformal transformations, the function has to have weight one half with respect to each variable. Thus, it can only be a function of the form

$$\hat{g}(d_{13}, d_{24}, \cdots) = \frac{1}{d_{13}} \frac{1}{d_{24}} \times (\cdots)$$
(5.23)

where the product runs over n distinct pairs of distances and each point i appears in one and only one of the terms. A distance between two points in the same bilocal is zero and cannot appear. So such terms are not present. In addition, the permutation symmetry under exchanges of B's and also of the two arguments of B imply that the overall coefficients of all terms are the same, up to disconnected terms, which are given by lower point functions of B's by cluster decomposition. For example, for a two point function of B's we have

$$\langle B(\underline{x_1, x_2}) B(\underline{x_3, x_4}) \rangle = \tilde{N} \left(\frac{1}{d_{13} d_{24}} + \frac{1}{d_{14} d_{23}} \right)$$
 (5.24)

The overall coefficients of the connected n point function of B's can be determined from the one in the n-1 point function by expanding one of the B's and looking at the stress tensor contribution, which is fixed by the stress tensor charge conservation identity. Therefore, all n point functions of B's are fixed up to a single constant, the constant in the two point function of the stress tensor, which is (up to a numerical factor) the same as \tilde{N} in (5.24).

We can consider a theory of N free bosons, with

$$B(\underline{x_1, x_2}) = \sum_{i=1}^{N} \underline{: \phi_i(x_1)\phi_i(x_2) :}$$
 (5.25)

where the :: imply that we do not allow contractions between these two ϕ fields. This theory has a higher spin symmetry and we also get the same correlators, except that the constant in front of the stress tensor and (5.24) is $\tilde{N} = N$. Another way to state the result is that all the correlators of the currents are the same as the ones we have in a theory of N bosons with N analytically continued, $N \to \tilde{N}$. However, as we argue below \tilde{N} should be an integer.

We did not show that the operators for the elementary fields (or partons) are present as good operators in the theory. It is clear that one cannot show that by looking purely at correlators on the plane, since one can project them out by imposing an SO(N) singlet condition which would leave all remaining correlators obeying the crossing symmetry relations, etc.

Let us discuss further lessons. First, we observe that the expansion of B's contains the currents with minus indices j_s . Thus, the three point function of B's contain the three point functions of currents with minus indices. All of these are the same as those of the free boson theory. Note that the possibility of odd three point functions has disappeared. Thus, we never needed to know much about the nature of the odd structures for the three point functions. From the OPE of two currents with minus indices we can get currents with arbitrary indices. Thus, from the six point function of B's we get the three point function of the currents for arbitrary indices, which coincides with the free boson answer. Similarly from a 2n point function of B's we get an n point function of currents with arbitrary indices which is equal to the free boson one for a theory of N bosons. By further performing OPE's we get other operators such as "double trace" or "double sum" operators. In conclusion, we have fixed all the correlation functions of the stress tensor, and also all the correlation function for all the higher spin conserved currents that appear in the j_2j_2 operator product expansion. These are currents that have even spins. In addition, we have also fixed the correlators of all other operators that appear in the operator product expansion of these operators. All such correlators are given by the corresponding correlators in a theory of N free bosons restricted to the O(N) invariant subsector.

We should remark that, though we made statements regarding currents that can be written as O(N) invariant bilinears, this does not mean that the theory is the "free O(N)" model. The theory can contain additional conserved currents (and still only one stress tensor). For example, for N=2M, we can consider M complex fields and restrict to the U(M) invariant sector. This theory obeys all the assumptions of our theorem. In particular, it still has a single conserved spin two current. However, we also have additional currents of odd spin. Of course, the currents with even spin that appear in the j_2j_2 OPE are still given by an O(2M) invariant combination of the fields and have the same correlators that we discussed above. Probably a little more work would show that if we had odd spin currents, they should also behave like those of free fields. Presumably one would construct a bilocal operator from the odd currents and argue as we did above. One could also wonder about other even spin currents which do not appear in the j_2j_2 OPE. Probably there cannot be such currents (with a single conserved spin two current), but we did not prove it.

Note that the correlation functions of stress tensors are all equal to the ones in the free theory. In particular, if we created a state with an insertion of the stress tensor at the origin we could compute the energy correlation functions that would be measured by idealized detectors (or calorimeters) at infinity. These are the energy correlation functions considered, for example in [3,4], and are computed by particular limits and integrals of correlation functions of the stress tensor. The n point energy correlator for a state created by the stress tensor is schematically $\langle 0|T^{\dagger}(q)\epsilon(\theta_1)\cdots\epsilon(\theta_n)T(q)|0\rangle$, where T(q) is the insertion of a stress tensor operator of four momentum (roughly) q at the origin and $\epsilon(\theta)$ are the energies per unit angle collected at ideal calorimeters sitting at infinity at the angle θ . These will give the same result as in the free theory. Namely, that the energy is deposited in two localized points, corresponding to the two partons hitting the calorimeters. This result is qualitatively similar to the Coleman-Mandula result for the triviality of the S-matrix. These energy correlation functions are infrared safe (or well-defined) observables which are, conceptually, rather close to the S-matrix. Here we see that these energy distributions are trivial.

5.5. Quantization of \tilde{N} : the case of bosons

The basic idea for showing that \tilde{N} is quantized uses the fact that the $N=\infty$ theory and the finite N theory have a different operator spectrum. The finite N spectrum is a truncation of the infinite N spectrum. This point was also emphasized recently in [29].

In order to prove the quantization of \tilde{N} we argue as follows. In the theory of N bosons we consider the operator

$$\mathcal{O}_q = \delta_{[j_1, \dots, j_q]}^{[i_1, \dots, i_q]} (\phi^{i_1} \partial \phi^{i_2} \partial^2 \phi^{i_3} \dots \partial^{q-1} \phi^{i_q}) (\phi^{j_1} \partial \phi^{j_2} \partial^2 \phi^{j_3} \dots \partial^{q-1} \phi^{j_q})$$
 (5.26)

where the δ function is the totally antisymmetric delta function of q indices. It is the object that arises we consider a contraction of two ϵ tensors of the from

$$\delta_{[j_1,\dots,j_q]}^{[i_1,\dots,i_q]} \propto \epsilon^{i_1,\dots,i_q,i_{q+1},\dots,i_N} \epsilon_{j_1,\dots,j_q,i_{q+1}\dots i_N}$$
(5.27)

This operator, (5.26), can be rewritten as sum of products of q bilinear operators. Once we have written it as a particular combination of bilinear operators we can consider any value of N and we can imagine doing analytic continuation in $N \to \tilde{N}$, with q fixed.

In particular, we can consider the norm of this state. We are interested in the \tilde{N} dependence of the norm of this state. We can show that

$$\langle \mathcal{O}_q \mathcal{O}_q \rangle = \tilde{N}(\tilde{N} - 1)(\tilde{N} - 2) \cdots (\tilde{N} - (q - 1))$$
(5.28)

where we only indicated the \tilde{N} dependence. Since we have q bilinears, the series expansion in $1/\tilde{N}$ has only q terms. In addition, the result should vanish for $\tilde{N}=1,2,\cdots,q-1$, and the leading power should be \tilde{N}^q .

Now, imagine that \tilde{N} was not integer. Then we could consider this operator for $q = [\tilde{N}] + 2$, where $[\tilde{N}]$ is the integer part of \tilde{N} . Then we find that (5.28) is

$$\langle \mathcal{O}_{[\tilde{N}]+2} \mathcal{O}_{[\tilde{N}]+2} \rangle = (\text{positive})(\tilde{N} - [\tilde{N}] - 1)$$
 (5.29)

where we only wrote the last term in (5.28), which is the only negative one. Thus, we have a negative norm state unless \tilde{N} is an integer. Therefore, unitarity forces \tilde{N} to be integer. Here we have phrased the argument in terms of the norm of a particular state, which might be changed by choosing a different normalization constant. However, we can get the same argument by consider the contribution to the OPE of a state like \mathcal{O}_q . If the norm is negative, then we will get a negative contribution to the OPE in the channel that is selecting this particular \mathcal{O}_q .

5.6. Fermionic-like bilocal operators

We now return to the second case in (5.11), where we have to use (5.15). We can now repeat the operations we did for the bosonic case in order to argue that

$$[Q_s, F_{-}(\underline{x}_1, \underline{x}_2)] = (\partial_1^{s-1} + \partial_2^{s-1})F_{-}(\underline{x}_1, \underline{x}_2)$$
(5.30)

The arguments are completely similar to the case of the boson. One needs to apply the same charge conservation identities. All the arguments are very similar, except that we now take the fermion like limit (5.1) and use the functions in (5.4). One can run over all the arguments presented for the boson-like limit and one can check that all the charge conservation identities have the same implications for the fermion-like case. This is shown in appendix J.

Once we show (5.30), we can now constrain the form of any correlator of the form $\langle F_{-}(\underline{x_1,y_1})\cdots F_{-}(\underline{x_n,y_n})\rangle$. Again it involves functions of differences of x_{ij}^- with each i appearing only in one argument of the function. In addition, we know that they should be rotational invariant and conformal covariant. However, as opposed to the bosonic case, in this case the conformal transformations of F_{-} are those of a product of fermions. We can take into account these transformations by using factors of fermion propagators, x_{ij}^+/d_{ij}^3 .

Together with permutation (anti) symmetry these constraints imply that the correlators are those of free fermions. For example, for the four point function we get

$$\langle F_{-}(\underline{x_1, x_2}) F_{-}(\underline{x_3, x_4}) \rangle = \tilde{N} \left(\frac{x_{13}^+}{d_{13}^3} \frac{x_{24}^+}{d_{24}^3} - \frac{x_{14}^+}{d_{14}^3} \frac{x_{23}^+}{d_{23}^3} \right)$$
 (5.31)

Similarly we can use the symmetries to fix all n point functions of F in terms of the single parameter \tilde{N} that appears in (5.31), or in the two point function of the stress tensor. For $\tilde{N} = N$, these n point functions agree with the ones we would obtain in a theory of N Majorana fermions with

$$F_{-}(\underline{x_1, x_2}) = \sum_{i=1}^{N} \underline{: \psi_{-}^{i}(x_1)\psi_{-}^{i}(x_2) :}$$
 (5.32)

The expansion of F_{-} contains all currents of twist one with minus indices. The further OPE of such currents contains twist one currents with other indices and also a scalar operator of twist two. In a theory of free fermions this is

$$\tilde{j}_0 = \sum_{i=1}^N \psi_+^i \psi_-^i \tag{5.33}$$

Thus, all the correlators of currents of even spin, plus the twist two scalar operator are fixed by the higher spin symmetry to be the same as in the theory of N free fermions.

5.7. Quantization of \tilde{N} : the case of fermions

One can wonder whether there exists any theory \tilde{N} is non-integer. Since \tilde{N} appears in the two point function of the stress tensor, we know that $\tilde{N} > 0$. We now argue for the quantization of \tilde{N} . The argument follows from considering the operator

$$\mathcal{O}_q \equiv : (\tilde{j}_0)^q : \tag{5.34}$$

For general \tilde{N} , this operator is defined by looking at the appropriate term in the operator product expansion of q \tilde{j}_0 operators that are coming together. And \tilde{j}_0 is itself also defined via a suitable operator product expansion of the original bilocals F_- . We can now compute the two point function of such operators, (5.34). More precisely, since the two point function is arbitrarily defined, we can look at the contribution to the OPE of the exchange of the operator \mathcal{O}_q . By the way, notice that the correlation function of any of the current bilinears or any product of such bilinears, is given by making an analytic continuation

in $N \to \tilde{N}$ of the free fermion results. It is important that when we make this analytic continuation, the number of currents in a correlator should be independent of N.

A crucial observation is that in a free fermion theory with N fermions we have that (5.34) is zero for q = N + 1. This means that such a contribution should not be present in the OPE.

Now, let us fix q and compute the two point function of \mathcal{O}_q for general \tilde{N} . (More precisely, we are talking about the contribution of this operator to the OPE). The expression has a $1/\tilde{N}$ expansion with precisely q terms, since that is the number of terms we get in a free fermion theory. We argue below that it should have the following \tilde{N} dependence

$$\langle \mathcal{O}_q \mathcal{O}_{\tilde{q}} \rangle = \tilde{N}(\tilde{N} - 1)(\tilde{N} - 2) \cdots (\tilde{N} - (q - 1))$$
 (5.35)

First, we see that (5.35) has precisely q terms. Second, notice that it should vanish for $\tilde{N} = 1, 2, \dots, q - 1$, since for such integer values, we have correlators identical to those of the free fermion theory where \mathcal{O}_q vanishes.

If \tilde{N} was not an integer, then we could set $q = [\tilde{N}] + 2$, where $[\tilde{N}]$ is the integer part of \tilde{N} . Then the norm (5.35) would be equal to

$$\langle \mathcal{O}_{[\tilde{N}]+2} \mathcal{O}_{[\tilde{N}]+2} \rangle = (\text{positive})(\tilde{N} - [\tilde{N}] - 1)$$
 (5.36)

where we only wrote the last factor in (5.35). The rest of the factors are positive. However, this last factor is negative if \tilde{N} is not an integer. Thus, we see that unitarity forces \tilde{N} to be an integer and we get precisely the same values as in the free fermion theory.

6. Arguments based on more generic three and four point functions

In this section we derive some of the above results in a conceptually more straightforward fashion, which ends up being more computationally intensive. We checked the statements below by using Mathematica. Of course, if we were to take light-like limits some of the computations simplify, and we go back to the discussion in the previous section.

6.1. Basic operations in the space of charge conservation identities

The presence of higher spin symmetries leads to charge conservation identities for three point functions which relate three point functions of conserved currents of different spins. Namely, we start from a three point function and demand that it is annihilated by Q_s . This imposes interesting constraints for the following reason. The three point functions have a very special form due to conformal symmetry and current conservation. The action of the higher spin charge (3.5) gives a linear combination of these three point functions and their derivatives. These derivatives do not commute with the action of the conformal group, so this single equation is equivalent to larger set of equations constraining the coefficients of the various three point functions. In other words, since conformal symmetry restricts the functional form of the three point functions, the single equation that results from Q_s charge conservation is typically enough to fix all the relative coefficients of the various three point functions that appear after we act with Q_s on each of the currents.

Let us first describe some operations that we can use over and over again to constrain the action of the symmetries.

Imagine that we are trying to constrain the action of Q_s on a current j_x and we would like to know whether current j_y is present or not in the transformation law

$$[Q_4, j_x] = j_y + \dots (6.1)$$

(generically, with some derivatives acting on j_y). Recall that through (3.6), then $[Q_4, j_y] = j_x + \cdots$. Now the basic charge conservation identity we can use is the one resulting from the action of Q_4 on $\langle j_2 j_x j_y \rangle$. First of all, notice that from the variation of j_2 we will necessarily get (see (2.4)) the term $\langle j_4 j_x j_y \rangle$ which must be non-zero if the j_x and j_y appear in each other transformations under Q_4 . The simplest possibility would be to find that the only solutions of the charge conservation identity are such that $\langle j_4 j_x j_y \rangle = 0$. Thus, our assumption about the presence of j_y in the variation of j_x was wrong. This is the basic operation of the elimination of j_y from $[Q_4, j_x]$.

If we can find that solutions of the charge conservation identity with $\langle j_4 j_x j_y \rangle \neq 0$ exist, this is consistent with the presence of j_y in the variation of j_x but does not necessarily imply it.

Another basic charge conservation identity operation can be used to the check that j_y is definitely present in the transformation of j_x . This is done via the charge conservation

identity resulting from the action of Q_4 on $\langle j_4 j_x j_x \rangle^{13}$. Notice that from the transformation of j_4 we will necessarily (see (2.4), (3.6)) get the term $\langle j_2 j_x j_x \rangle$ which must be non-zero due to the fact that j_2 generates conformal transformations. Now imagine that for the solution of the charge conservation identity to exist the term $\langle j_4 j_x j_y \rangle$ should be necessarily non-zero. This means that j_y is necessarily present in the transformation of j_x .

Very often using these two operations allow us to fix completely which operators do appear in the transformation of the given conserved current.

6.2. From spin three to spin four

In this section we show that in any theory that contains a conserved current of spin three then the conserved spin four current is necessarily present.

Let us first consider the most general CFT that has spin three current. We have that the most general transformation has the form

$$[Q_3, j_2] = \alpha_0 \partial^4 j_0 + \alpha_1 \partial^3 j_1 + \alpha_2 \partial^2 j_2 + \alpha_3 \partial j_3 + \alpha_4 j'_4,$$

$$[Q_3, j_3] = \beta_1 \partial^4 j'_1 + \beta_2 \partial^3 j_2 + \beta_3 \partial^2 j'_3 + \beta_4 \partial j_4 + \beta_5 j_5.$$
(6.2)

Let's make several comments on this expression. Primes stand for the fact that the same or a different current of the same spin can, in principle, appear in the right hand side. Notice that $\alpha_4 = 0$ since otherwise Q_3 is not translation invariant. Integrating both sides we would get $[Q_3, P_-] = Q_4$, which would mean that Q_3 is not translation invariant, in contradiction with (2.3) and current conservation. From (2.4), α_3 should be non-zero, and so is β_2 .

As the next step we consider the charge conservation identity obtained by Q_3 acting on $\langle j_2 j_2 j_3 \rangle$

$$0 = \langle [Q_3, j_2(x_1)j_2(x_2)j_3(x_3)] \rangle = 0$$
(6.3)

This is essentially the same charge conservation identity we considered in section five. The solution exists only for $\beta_4 \neq 0$. In other words, the spin four current is necessary present in the theory.

Another feature of this exercise that is worth mentioning is that the general solution of the charge conservation identities involves three distinct pieces

$$\langle jjj\rangle = \langle jjj\rangle_{boson} + \langle jjj\rangle_{fermion} + \langle jjj\rangle_{odd}$$
 (6.4)

In practice, we first check that $[Q_4, j_2] \sim \partial^3 j_2 + \cdots$ and then we use $\langle j_2 j_x j_x \rangle$.

the first two pieces correspond to the free boson and free fermion three point functions respectively. We expect these solutions to be present in the theory of complex boson and fermion. The odd piece does not come from any of free theories and is parity violating ¹⁴. We will elaborate the nature of the odd piece later. But we should emphasize that at the level of three point functions one can find three independent solutions of higher spin charge conservation identities ¹⁵.

After an illustration of this particular example we can formulate the general recipe. Assume that the theory has a conserved current j_s of spin $s \neq 4$ and s > 2. We again consider the charge conservation identity obtained by Q_s acting on $\langle j_2 j_2 j_s \rangle$ and arrive at the conclusion that j_4 must be present in the spectrum. This type of charge conservation identity is especially simple to analyze using the light cone limit described in the previous section.

So that we take it as a given that we always have a spin four current.

6.3. Analysis of three point functions using the spin four current.

In this section we look in detail at the action of the Q_4 charge. Again we present only the results of the computations which are straightforward but tedious. We can argue that

$$[Q_4, j_2] = \partial^5 j_0 + \partial^3 j_2 + \partial j_4 \tag{6.5}$$

We have eliminated j_1 and j_3 by considering the charge conservation identity corresponding to the action of Q_4 on $\langle j_2 j_2 j_1 \rangle$ and $\langle j_2 j_2 j_3 \rangle$. We used general transformation laws for j_1 , j_3 (3.5). Then we consider the action of Q_4 at $\langle j_2 j_2 j_2 \rangle$ and use that we have already shown that $\langle j_2 j_2 j_4 \rangle$ is nonzero. The stress tensor three point function can have three different pieces

$$\langle j_2 j_2 j_2 \rangle = \langle j_2 j_2 j_2 \rangle_{boson} + \langle j_2 j_2 j_2 \rangle_{fermion} + \langle j_2 j_2 j_2 \rangle_{odd}$$

$$(6.6)$$

The charge conservation identity gives three different solutions involving other spins corresponding to these three different pieces. In other words, the charge conservation identity equations for the boson, fermion and odd pieces do not mix. However, if the boson or odd pieces in (6.6) are non-zero then the charge conservation identity implies that the current j_0 exists and appears as in (6.5). The fermion solution does not require a j_0 .

 $^{^{14}}$ Since it gives odd contribution to the three point function of stress tensors.

¹⁵ We will show below that the odd solutions appear in the theories with higher spin symmetry broken at $\frac{1}{N}$ order.

If the stress tensor is unique, then the fact that j_0 exists, implies that $\langle j_2 j_2 j_2 \rangle_f = 0$. This is done by considering the Q_4 charge conservation Identity for $\langle j_0 j_2 j_2 \rangle$. Q_4 on j_0 gives j_2 due to (3.6) and (6.5). The non-zero fermion part of these three point functions is not compatible with this charge conservation identity.

Then the problem separates into two problems. First we consider the case where $\langle j_2j_2j_2\rangle_f=0$ and then the case when $\langle j_2j_2j_2\rangle_b=0$. Both cannot be zero due the fact that j_2 generates conformal transformations for j_2 , and the fact that the odd piece does not contribute to the action of conserved charges¹⁶. The odd piece can be eliminated at this point by an energy correlation argument, as shown in appendix C. But the reader not familiar with that technique can wait a little longer until we eliminate it in a more straightforward way.

6.4. Constraining the four point function of j_0

First we can constrain the action of Q_4 on j_0 . By a method very similar to the one we used for the stress tensor we can prove that

$$[Q_4, j_0] = \partial^3 j_0 + \gamma \partial j_2 \tag{6.7}$$

where we wrote the constant γ appearing in the transformation explicitly. Again we eliminate j_1 and j_3 by considering the Q_4 charge conservation identities on $\langle j_2 j_1 j_0 \rangle$ and $\langle j_2 j_3 j_0 \rangle$. The other two terms can be found from the charge conservation identity for Q_4 acting on $\langle j_0 j_2 j_2 \rangle$, and using (6.5) and the fact that $\langle j_0 j_0 j_2 \rangle$ is nonzero.

Now we consider Q_4 acting on the four point function $\langle j_0 j_0 j_0 j_0 \rangle$. This gives

$$\partial_1^3 \langle j_0 j_0 j_0 j_0 \rangle + \gamma \partial_1 \langle j_2 j_0 j_0 j_0 \rangle + [1 \leftrightarrow 2] + [1 \leftrightarrow 3] + [1 \leftrightarrow 4] = 0$$
 (6.8)

In order to solve this we first need to write the most general four point function with one insertion of the stress tensor and three scalars, $\langle j_2 j_0 j_0 j_0 \rangle$. This can be done using the techniques described in [27,28]. There are two possible forms, one is parity even and the other is parity odd. The equation (6.8) splits into two one for the parity even and the

Consider $\int dx^- dy \langle j_{s_1}(x)j_{s_2}(x_2)j_{s_3}(x_3)\rangle_{odd}$ with $y_2 = y_3 = 0$ then under $y \to -y$ the integrand is odd and, thus, the integral vanishes.

other for the parity odd piece. The general form for the parity even piece involves certain conformal invariants Q_{ijk} constructed out of the positions and the polarization tensors

$$\langle j_2(x_1)j_0(x_2)j_0(x_3)j_0(x_4)\rangle = \frac{Q_{123}^2g(u,v) + Q_{124}^2g(v,u) + Q_{134}^2\tilde{g}(u,v)}{x_{12}^2x_{34}^2}$$

$$\tilde{g}(u,v) = \frac{1}{u}g(\frac{v}{u},\frac{1}{u})$$
(6.9)

where the relevant conformal invariants simplify to

$$Q_{ijk} = \frac{x_{i,j}^{+}}{x_{i,j}^{2}} - \frac{x_{i,k}^{+}}{x_{i,k}^{2}} , \qquad u = \frac{x_{13}^{2} x_{24}^{2}}{x_{12}^{2} x_{34}^{2}} , \qquad v = \frac{x_{14}^{2} x_{23}^{2}}{x_{12}^{2} x_{34}^{2}}$$
 (6.10)

because we consider only the minus polarization. We define f(u,v) via

$$\langle j_0 j_0 j_0 j_0 \rangle = \frac{f(u, v)}{x_{12}^2 x_{34}^2}$$

$$f(u, v) = f(v, u) = \frac{1}{v} f(\frac{u}{v}, \frac{1}{v})$$
(6.11)

Inserting (6.11) and (6.9) into (6.8), we get an equation which depends both on the cross ratios and the explicit points x_i . By applying conformal transformations, and a lot of algebra, we can get a set of equations purely in terms f, g (and u and v). There is a solution where g = 0, which has a factorized dependence of the coordinates. There is also a solution with both g and f. The sum of these two solutions is

$$f(u,v) = \alpha \left(1 + \frac{1}{u} + \frac{1}{v}\right) + \beta \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{v}} + \frac{1}{\sqrt{uv}}\right),$$

$$\gamma g(u,v) = \beta \frac{9}{20\sqrt{u}}.$$
(6.12)

By taking OPE of $\langle j_0 j_0 j_0 j_0 \rangle$ we can extract, for example, the $\langle j_0 j_0 j_0 \rangle$ structure constant. We can fix γ then by considering the j_2 charge conservation identity. In other words, integrating j_2 to get the action of P_- on $\langle j_0 j_0 j_0 \rangle$, for example.

These are the free field theory correlators. The term proportional to α is the disconnected contraction and the one involving β is the connected one. In a theory of N free scalars we can set $\alpha = 1$ by a choice of normalization for the operators. Then $\beta \sim 1/N$. Notice that we were able to fix two four point functions using just one charge conservation identity.

6.5. No parity odd piece

While it is clear that we cannot write an odd piece for the four point function of scalars we can do it for $\langle j_2 j_0 j_0 j_0 \rangle$. The unique structure in this case takes the following form in the embedding formalism (see [28] for conventions)

$$\langle j_2(x_1)j_0(x_2)j_0(x_3)j_0(x_4)\rangle \sim \frac{\epsilon(Z_1, P_1, P_2, P_3, P_4)}{(P_1P_2)^2(P_1P_3)^{3/2}(P_1P_4)^{3/2}(P_3P_4)^{1/2}}$$

$$[Q_{123}g_1(u, v) + Q_{134}g_2(u, v) + Q_{142}g_3(u, v)]$$
(6.13)

Inserting this in the charge conservation identity (6.8), $\sum \partial \langle j_2 j_0 j_0 j_0 \rangle = 0$ we find that there is no solution.

By taking the OPE of $\langle j_2 j_0 j_0 j_0 \rangle$ we can concentrate on the twist one channel where the stress tensor is propagating. The relevant three point functions are $\langle j_2 j_0 j_2 \rangle$ and $\langle j_2 j_0 j_0 \rangle$.

Both of them are non-zero. The function $\langle j_2 j_0 j_0 \rangle$ is non-zero because stress tensor generates conformal transformation and $\langle j_2 j_0 j_2 \rangle$ is non-zero because of the Q_4 charge conservation identity for $\langle j_2 j_2 j_2 \rangle$.

Due to the triangle inequality (4.2) the $\langle j_2 j_0 j_2 \rangle$ is the only odd three point function that is non-zero in the twist one sector OPE expansion of $\langle j_2 j_0 j_0 j_0 \rangle$.

The fact that the four point function does not have the odd piece forces us to set the odd piece of $\langle j_2 j_0 j_2 \rangle$ to zero. Then through the higher spin charge conservation identities we will set the odd part of the whole tower of conserved currents three point functions to zero. Thus, while the odd pieces of three point functions respect higher spin symmetry, at the level of four point functions they are eliminated in the theories where higher spin symmetry is exact.

6.6. Case of fermions

The above discussion can be repeated for the case that $\langle j_2 j_2 j_2 \rangle_f \neq 0$. In this case we do not expect a twist one, spin zero field. Here we can show that

$$[Q_4, j_2] = \partial^3 j_2 + \partial j_4 \tag{6.14}$$

In principle, we could act with Q_4 on the four point function $\langle j_2 j_2 j_2 j_2 \rangle$ and use (6.14) to fix it completely. Though this probably works, we have not managed to do it due to the large number of conformal structures that are possible.

Instead, one can take a longer route by first showing that a certain twist two scalar operator \tilde{j}_0 exists, finding its transformation laws under Q_4 and then showing that its four point function is the same as that of $\tilde{j}_0 = \epsilon^{\alpha\beta}\psi_\alpha\psi_\beta$ in a theory of free fermions.

In this section we consider not just the minus component of the currents but also the third, or perpendicular component. For the charge Q_4 we continue to focus on the all minus component.

We consider the charge conservation identity from the action of Q_4 on $\langle j_2 j_2 j_{2-\perp} \rangle$.

The new ingredient is the variation $[Q_4, j_{-\perp}]$ which should have twist two operators in the right hand side. We write the most general form of this variation denoting by \tilde{j}_s a twist two operator of spin s (and without tilde for the twist one ones)

$$[Q_4, j_{2\perp}] = \sum_{i=0}^{4} \partial^{4-i} (\tilde{j}_i + \partial_{\perp} j_i + j_{(i+1)\perp})$$
(6.15)

where all implicit tensor indices are minus.

Now as we are having in mind the Q_4 charge conservation identity for $\langle j_2 j_2 j_{2\perp} \rangle$ we can ignore all the odd spin twist one current due to (4.3). Moreover, we can fix \tilde{j}_{4-} to zero using the translation invariance argument. Also we know that for the fermion solution j_0 term is absent. And here we are interested in the fermion part of the solution for $\langle j_2 j_2 j_2 \rangle$. One can be confused by the lack of perpendicular index in the twist two sector. It comes from an ϵ tensor as $\epsilon_{-\perp\mu}\partial^{\mu}=\epsilon_{-\perp+}\partial_{-}$, so that with more general indices we have

$$[Q_4, j_{2\mu\nu}] = \partial_-^2 [\partial_\mu \epsilon_{-\nu\sigma} + \partial_\nu \epsilon_{-\mu\sigma}] \partial_{\sigma} \tilde{j}_0 + \dots$$
 (6.16)

This epsilon tensor implies that in the charge conservation identity $\langle j_2 j_2 j_{2-\perp} \rangle_{even}$ should cancel the odd contribution $\langle j_2 j_2 \tilde{j}_0 \rangle_{odd}$. The relevant terms that we need for the charge conservation identity are

$$[Q_4, j_{2\perp}] = \partial^4 \tilde{j}_0 + \partial^2 \tilde{j}_2 + \partial^2 \partial_{\perp} j_2 + \partial_{\perp} j_4 + \partial^3 j_{2\perp} + \partial j_{4\perp} + \dots$$

$$(6.17)$$

The dots denote terms that do not have any overlap with j_2j_2 . So we end up with the following charge conservation identity to be checked. Here we think about all even parts as being generated by fermion three point functions

$$\langle [\partial^{3} j_{2}] j_{2} j_{2\perp} \rangle_{even} + \langle [\partial j_{4}] j_{2} j_{2\perp} \rangle_{even} + [1 \leftrightarrow 2] = \langle j_{2} j_{2} [\partial^{4} \tilde{j}_{0}] \rangle_{odd} + \langle j_{2} j_{2} [\partial^{2} \tilde{j}_{2}] \rangle_{odd} + \langle j_{2} j_{2} [\partial^{2} \partial_{(-} j_{2\perp})] \rangle_{even} + \langle j_{2} j_{2} [\partial_{(-} j_{4\perp})] \rangle_{even}.$$

$$(6.18)$$

The only new three point functions involved are $\langle j_2 j_2 \tilde{j}_0 \rangle_{odd}$ and $\langle j_2 j_2 \tilde{j}_2 \rangle_{odd}$. Both are fixed by conformal symmetry, and current conservation for j_2 , up to a constant. The charge conservation identity implies that the twist two scalar \tilde{j}_0 needs to exist and it needs to appear in the right hand side of

$$[Q_4, j_{2\perp}] = \partial^4 \tilde{j}_0 + \partial^3 j_{2\perp} + \partial j_{4\perp} + \dots \tag{6.19}$$

By performing an analysis similar to what was done for the bosonic case one can show that nothing else can appear in the right hand side of (6.19). In addition, one can then constrain the action of Q_4 on \tilde{j}_0 and find

$$[Q_4, \tilde{j}_0] = \partial^3 \tilde{j}_0 + \partial \partial_{[-} j_{2\perp]}$$

$$(6.20)$$

This again requires looking at many charge conservation identities to eliminate all the other possible terms that could appear. Again, notice that the second term involves an ϵ tensor.

After this, one can look at the charge conservation identity for the four point function $\langle \tilde{j}_0 \tilde{j}_0$

7. Higher spin symmetries broken at order $\frac{1}{N}$

So far we considered the case where the symmetries are exactly conserved. There are some interesting theories, mainly large N O(N), Sp(N), U(N) vector models where the symmetries are "almost" conserved. By this we mean that the anomalous dimensions of the currents are of order 1/N. Furthermore the correlators of the theory have 1/N expansion. We will make now some remarks about this case here.

To explain this imagine a situation when the higher spin currents have anomalous dimension $\Delta = s + 1 + O(\frac{1}{N})$. The operators have a single "trace" vs. a multitrace structure, and to leading order in N the correlators of multitrace operators factorize¹⁷.

Rather than giving a general discussion, here we will focus on one specific case that will illustrate the method, leaving a more general discussion for the future.

¹⁷ We should rather speak of a "single sum" vs. a "multi-sum" structure.

We can illustrate the method by considering a large N SO(N) Chern-Simons matter theory with an SO(N) level k Chern-Simons action coupled to N Majorana fermions. This theory was considered recently in $[30]^{18}$. In this theory, in addition, we have a coupling $\lambda \sim N/k$ which is very small when the Chern-Simons level is very large, $k \gg N$. So we now have this second expansion parameter which we will also use.

In the limit that $\lambda = 0$ we have the theory of free fermions. The "single trace" operators in this theory are spanned by the twist one currents j_s and the twist two pseudoscalar operator \tilde{j}_0 . Let us normalize them so that their two point functions are one. Let us consider the most general form of the divergence of the J_4 current in this theory

$$\partial_{\mu} J^{\mu}_{---} = \frac{1}{\sqrt{N}} \left[a_1 \partial_{-} \tilde{j}_0 j_2 + a_2 \tilde{j}_0 \partial_{-} j_2 \right]. \tag{7.1}$$

The divergence should be twist three operator with spin three. The terms in the right hand side are all the operators we can write down in this theory. Note that there are no single trace primaries that can appear. Notice that at this point we do not use any information about the microscopic theory except the spectrum at $N = \infty$.

We can now consider partially broken charge conservation identities, by integrating an insertion of (7.1) in a correlation function

$$\int d^3x \langle \partial_{\mu} J^{\mu}_{---} \mathcal{O}_1 \cdots \mathcal{O}_n \rangle = \sum_i \int_{S_i} \langle j^n_{---} \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \tag{7.2}$$

Here we have used Stoke's theorem to integrate the divergence on a region which is the full R^3 minus little spheres around the insertion of each operator. The surface terms have the usual expression for the charges in terms of the currents. Except that now they are not conserved. However, since the non-conservation is a small correction, we can act on each of the operators by the charges, up to 1/N corrections. In the left hand side we can insert the expression (7.1) and use the leading N, factorized correlator. In our normalization the action of the "charges", have a $1/\sqrt{N}$, so that left hand side and right hand side of (7.2) are of the same order.

As a first example, consider the case where the operators are simply $\mathcal{O}_1\mathcal{O}_2 = \tilde{j}_0 j_2$. In this case, the charges, to leading order annihilate the correlator in a trivial way, using

¹⁸ Actually, they considered a U(N) gauge group, but the story is very similar. For a similar theory with scalars see [31].

(6.20), (6.14). All that remains is the integral of the right hand side of (7.1). In order for this to vanish, we need that

$$a_2 = -\frac{2}{5}a_1. (7.3)$$

So this relative coefficient is fixed in this simple way, for all λ , to leading order in 1/N. This is a somewhat trivial result since it also follows from demanding that the special conformal generator K^- annihilates the right hand side of (7.1). We have spelled it out in order to illustrate the use of the broken symmetry.

As a less trivial example, consider the insertion of the same broken charge conservation identity in the three point function of the stress tensor. We will do this to leading order in λ . We get

$$\sum_{i} \int_{S_{i}} \langle j_{---}^{n} j_{2}(x_{1}) j_{2}(x_{2}) j_{2}(x_{3}) \rangle \sim \frac{a_{1}}{\sqrt{N}} \int d^{3}x \langle [\partial \tilde{j}_{0} j_{2} - \frac{2}{5} \tilde{j}_{0} \partial j_{2}](x) j_{2}(x_{1}) j_{2}(x_{2}) j_{2}(x_{3}) \rangle.$$
(7.4)

Now let's take the large N limit in this equation. In the left hand side we can substitute the action of the charges on each of the operators. This gives

$$\langle [Q_4, j_2 j_2 j_2] \rangle \sim \frac{1}{\sqrt{N}} \left(\langle \partial^3 j_2 j_2 j_2 \rangle + \langle \partial j_4 j_2 j_2 \rangle + \text{action on the other } j_2' \text{s} \right)$$
 (7.5)

Notice that this is of order $\frac{1}{N}$.

In the right hand side of (7.4) the order $\frac{1}{N}$ terms come from

$$\int d^3x \langle \partial \langle j_2(x)j_2(x_1) \rangle \rangle \langle \tilde{j}_0(x)j_2(x_2)j_2(x_3) \rangle \sim \partial_1^5 \int d^3x \frac{1}{|x_1 - x|} \langle \tilde{j}_0(x)j_2(x_2)j_2(x_3) \rangle \quad (7.6)$$

where we have integrated by parts and used that all indices are minus $(\partial_1 = \partial_{x_1^-})$.

Now the final result of the integral in the right hand side of (7.6) is the same as the three point function $\langle j_0 j_2 j_2 \rangle$. Namely, the three point function involving a twist *one* scalar, as opposed to the twist two scalar \tilde{j}_0 that we started with. This can be seen by the fact that the integral in (7.6) (before taking the ∂_1^5 derivative) has the same conformal properties as $\langle j_0 j_2 j_2 \rangle$.

If the current were exactly conserved, we would set (7.5) to zero. In that case, the only solution is the one corresponding to the free fermion structure in the three point functions. The reason is that the free boson, or odd solutions of this charge conservation identity require a twist one operator, j_0 . Recall that we had said in section six that these three point function ward identities have three independent solutions involving only the fermion,

or only the boson or only the odd structures. However, in (7.4) something remarkable has happened. The right hand side, which comes from the lack of conservation of the current, is mocking up perfectly the contribution we would have had in a theory with a twist one scalar. This allows for more nontrivial solutions which can be the superposition of all three structures for the three point functions.

Notice that if we consider now the charge conservation identity expanded in the 't Hooft parameter λ , then the λ^0 term will satisfy free fermion result. At the λ^1 the integral in (7.6) generates a result equal to the *odd* structure for $\langle j_0 j_2 j_2 \rangle$. Thus, at order λ^1 , the odd structure for $\langle j_2 j_2 j_2 \rangle$ is generated.

We expect that further analysis along the lines explained above should fix all the leading N three point functions, exactly in λ .

We expect to be able to apply a similar ideas to the interacting fixed point of O(N) model. In this case the spectrum of leading N theory is the same, except that the twist two operator is parity even.

Finally, let us comment on the case of weakly coupled gauge theories with matter in the adjoint. In this case we will typically have single trace operators that can, and will, appear in the right hand side of the divergence of the currents, once we turn on the coupling. In this case, we also expect to be able to use the broken charge conservation identities to analyze the theory, though it is not clear whether this will be any simpler than applying usual perturbation theory. One feature is that we will be dealing purely in terms of observables, without ever discussing things like "gauge fixing", etc.

8. Case of two conserved spin two currents

In this section we relax the assumption of a single stress tensor and we generalize the discussion to the case of two stress tensors. Presumably something similar will hold for more than two, but we have not studied it in detail.

We now consider a CFT that has exactly two symmetric traceless conserved spin two currents. One of them generates conformal transformations and we denote it by j_2 . The usual minus generator built out of it is translation along the minus direction $Q_2 = P_-$. Another current we denote as \hat{j}_2 and the correspondent charge would be $\hat{Q}_2 = \hat{P}_-$. We assume that these two currents are orthogonal and we normalize their two point functions as

$$\langle j_{2-}(x_1)j_{2-}(x_2)\rangle = \langle \hat{j}_{2-}(x_1)\hat{j}_{2-}(x_2)\rangle = c\frac{(x^+)^4}{(x^2)^{d+2}}, \qquad \langle j_{2-}(x_1)\hat{j}_{2-}(x_2)\rangle = 0 \quad (8.1)$$

here we used the freedom to rescale \hat{j}_2 .

The most general form of the transformation consistent with the conformal properties of generators, two point function charge conservation identities and non-zero three point functions is

$$[P_{-}, j_{2}] = \partial_{-}j_{2}, [P_{-}, \hat{j}_{2}] = \partial_{-}\hat{j}_{2}$$

$$[\hat{P}_{-}, j_{2}] = a \,\partial^{2}j_{1} + \partial\hat{j}_{2} (8.2)$$

$$[\hat{P}_{-}, \hat{j}_{2}] = \partial j_{2} + b \,\partial\hat{j}_{2}$$

We then consider the $\langle [\hat{P}_-, j_2 j_2 \hat{j}_2] \rangle = 0$ charge conservation identity, which sets a = 0.

Now, for any b we can introduce a new basis

$$J_{2} = \frac{1}{2} \left(1 + \frac{b}{\sqrt{b^{2} + 4}} \right) j_{2} - \frac{1}{\sqrt{b^{2} + 4}} \hat{j}_{2}$$

$$\hat{J}_{2} = \frac{1}{2} \left(1 - \frac{b}{\sqrt{b^{2} + 4}} \right) j_{2} + \frac{1}{\sqrt{b^{2} + 4}} \hat{j}_{2}$$
(8.3)

such that the commutation relations take the form

$$[Q_{J_2}, J_2] = \partial J_2, [Q_{\hat{J}_2}, \hat{J}_2] = \partial \hat{J}_2, [Q_{\hat{J}_2}, J_2] = 0, [Q_J, \hat{J}_2] = 0.$$
(8.4)

In this new basis there are two orthogonal $\langle J_2 \hat{J}_2 \rangle = 0$ conserved spin two currents such that each of them generates translation for itself and leaves intact the other one.

We can now consider the correlation function with J_2 's, \hat{J}_2 's and an insertion of $e^{iQ_{J_2}a}$

$$\langle \hat{J}_{2}(y_{1})...\hat{J}_{2}(y_{n})J_{2}(x_{1})...J_{2}(x_{n})\rangle = \langle e^{iQ_{J_{2}}a}\hat{J}_{2}(y_{1})...\hat{J}_{2}(y_{n})J_{2}(x_{1})...J_{2}(x_{n})e^{-iQ_{J_{2}}a}\rangle$$

$$= \langle \hat{J}_{2}(y_{1})...\hat{J}_{2}(y_{n})e^{iQ_{J_{2}}a}J_{2}(x_{1})...J_{2}(x_{n})e^{-iQ_{J_{2}}a}\rangle$$
(8.5)

where we used the fact that $[Q_{\hat{J}_2}, J_2] = 0$. Then using the fact that $[Q_{J_2}, P_-] = 0$ and also the fact that $[Q_{J_2}, J_2] = [P_-, J_2]$ we can rewrite (8.5) as

$$\langle \hat{J}_{2}(y_{1})...\hat{J}_{2}(y_{n})J_{2}(x_{1})...J_{2}(x_{n})\rangle = \langle \hat{J}_{2}(y_{1})...\hat{J}_{2}(y_{n})e^{iP_{-}a}J_{2}(x_{1})...J_{2}(x_{n})e^{-iP_{-}a}\rangle$$

$$= \langle \hat{J}_{2}(y_{1})...\hat{J}_{2}(y_{n})J_{2}(x_{1}+a^{-})...J_{2}(x_{n}+a^{-})\rangle.$$
(8.6)

where all J_2 's are translated along the minus direction. By the cluster decomposition assumption we get¹⁹

$$\langle \hat{J}_2(y_1)...\hat{J}_2(y_n)J_2(x_1)...J_2(x_n)\rangle = \langle \hat{J}_2(y_1)...\hat{J}_2(y_n)\rangle\langle J_2(x_1)...J_2(x_n)\rangle. \tag{8.7}$$

Let us make this more clear. We first choose the x_i and y_j to be spacelike separated and such that as $a \to +\infty$ the distances between any x_i and any y_j grow. One can check that such a choice of points is possible, and it still allows us to move all the points in a small neighborhood without violating this property. Once we establish the decoupling for such points, we can analytically continue the result for all points.

Thus, these correlators are completely decoupled. Notice that, in particular, if we consider collider physics observables like energy correlation functions for the "energies" defined from J_2 or \hat{J}_2 , we would obtain completely decoupled answers. However, we cannot conclude at this point that we have two decoupled theories, since we could still consider two theories that share a global symmetry and impose a singlet constraint on the operators. However, it is clear that in some sense, the two theories are dynamically decoupled. All that we have said so far is valid for any theory with two spin two conserved currents, independently of whether we have any higher spin generator.

8.1. Adding higher spin conserved currents

Now we consider a theory with a conserved higher spin current, j_s , and exactly two stress tensors. We know that $[Q_s, j_2] \sim j_s$, which implies that $[Q_s, j_s] = j_2 + \cdots$. Thus, the right hand side of $[Q_s, j_s]$ has either J_2 or \hat{J}_2 , or both. Let us first assume that it has J_2 . Then the Q_s ward identity on $\langle J_2 J_2 j_s \rangle$ implies that there is an infinite number of currents in $\underline{J_2 J_2}_b$ or in $\underline{J_2 J_2}_f$. Now, nothing that appears in the right hand side of these light cone limits can have any non-zero correlator with \hat{J}_2 , due to (8.7). Thus, from this point onwards, the analysis is effectively the same as if we had only one stress tensor. Of course, the same holds if \hat{J}_2 appears in the right hand side of $[Q_s, j_s]$.

One simple example with two conserved spin two currents is a free $\mathcal{N}=1$ supersymmetric theory with N bosons and N fermions, with an O(N) singlet constraint. The two conserved spin two currents are the stress tensor of the boson and the one of the fermions.

The situation is more subtle when we have more than two spin two conserved currents. For example, in a theory with two free bosons, ϕ_1 , ϕ_2 , we have three conserved spin two conserved currents, schematically $\partial \phi_1 \partial \phi_1$, $\partial \phi_1 \partial \phi_2$ and $\partial \phi_2 \partial \phi_2$. Clearly in this case the theory decouples into two theories, and not into three!

9. Conclusions and discussion

In this article we have studied theories with exactly conserved higher spin currents, with spin s > 2. We have shown that all correlators of the currents and the stress tensor are those of a free theory. More precisely, we have made the technical assumption of a single spin two current. Under this assumption, the only two cases are those of a free scalar or a free fermion. More precisely, we proved that the correlators of the currents are

the same as in the theory of free bosons or free fermions, but we did not demonstrate the existence of a free scalar or a free fermion operator.

This is in the same spirit as the Coleman-Mandula theorem, [1,2], extended here to theories without an S-matrix.

It can also be viewed as a simple exercise in the bootstrap approach to field theories. One simply starts with the currents, constructs the symmetries, and one ends up fixing all the correlation functions. One never needs to say what the Lagrangian is. Of course, the answer is very simple, because we end up getting free theories!

In this paper we considered correlators of stress tensors and currents but not of other operators that the theory can have. In two dimensions, one can find explicitly the correlators of stress tensors, but that does not mean that the theory is free. On the other hand, in three dimensions we expect that the simple form of correlators that we obtained will imply that the theory is indeed free. In fact, one can consider the conformal collider physics observables that come from looking at energies collected by "detectors" at infinity for a state created by the stress tensor [3,4]. The n point energy correlator for a state created by the stress tensor is schematically $\langle 0|T^{\dagger}(q)\epsilon(\theta_1)\cdots\epsilon(\theta_n)T(q)|0\rangle$, where T(q) is the stress tensor insertion with four momentum (roughly) q at the origin and $\epsilon(\theta)$ are the energies per unit angle collected at ideal calorimeters sitting at infinity situated at the angle θ . These can be computed by considering n+2 correlation functions of stress tensors. Since these agree with the ones in the free theory, we expect that the theory is free. Notice also that these infrared safe observables are conceptually rather close to the S-matrix, since they are measured at infinity in Minkowski space. Nevertheless, one would like to be able to understand directly the constrains of the higher spin symmetry on other operators and their correlators. For example, it is natural to conjecture that their dimensions are integer or half integer (if the stress tensor is unique). As an example of a theory that contains an extra operator, we can consider N fermions with an SO(N) singlet constraint and the operator $\epsilon^{i_1,\dots,i_N}\psi_-^{i_1}\dots\psi_-^{i_N}$. This can certainly not be produced from the currents when N is odd. For N odd this has a half integer dimension.

It should be simple to extend these results to four dimensional field theories. In fact, our approach that uses the light-like limits in section five is expected to work with few modifications. We expect that we will need light-like limits that pick out the free boson, free fermion and free Maxwell fields. In higher dimensions we can have other free fields, such as the self dual tensor in six dimensions.

Recently there have been many studies of a four dimensional higher spin gravity theory proposed by Vasiliev ([8] and references therein). These theories have higher spin gauge symmetry in the bulk. They also have an AdS_4 vacuum solution. If one chooses AdS boundary conditions that preserve the higher spin symmetry, then our methods show that the theory is essentially the same as a free theory on the boundary. Thus, we have proven the conjecture in [10] for the free case. More concretely it is the theory of N free bosons or free fermions with an O(N) singlet constraint. Given that the free case works, then [14] showed that the conjecture followed for the interacting case. Other arguments were presented in [15,16]. If the boundary conditions break the higher symmetry in a slight way, then one can also use the symmetries to constrain the results. Notice also that our results for the quantization of \tilde{N} show that the coupling constant in a unitary Vasiliev theory with higher spin symmetric boundary condition is quantized.

It is also interesting that theories with almost conserved higher spin currents can be analyzed in this spirit. We discussed some simple computations in the case of large N vector models (recently studied in [30],[31]). Here one can use the slightly broken symmetries to get interesting results about the correlators. Of course, our previous analysis of the exact case is the backbone of the analysis for the slightly broken one. We suspect that correlators in these theories can be completely fixed by these considerations alone, and it would be nice to carry this out explicitly.

Notice that for any weakly coupled theory we have a slightly broken higher spin symmetry. In general, the breaking of the higher spin symmetry can occur at the single trace level. This occurs, for example, in gauge theories with adjoint fields. One might be able to perform perturbative computations using the higher spin currents and the pattern of symmetry breaking. Of course, one would recover the results of standard perturbation theory. However, the fact that we work exclusively with gauge invariant physical observables might lead to important simplifications, particularly in the case of gauge theories. Notice, by the way, that the bilocal operators that we have introduced in (5.12) (5.30) are simply the (Fourier transform) of the operators whose matrix elements are the parton distribution functions. In an interacting theory, these operators also have a Wilson line connecting the two "partons".

More ambitiously, one would like to understand large N limits of theories with adjoints, such as $\mathcal{N}=4$ super Yang Mills, in the 't Hooft limit. In general, the single trace higher spin currents can have large anomalous dimensions compared to one. However, the anomalous dimensions are still small compared to N. It is natural to wonder whether

they continue to impose interesting constraints. This is closely related to the possible emergence of a useful higher spin symmetry in string theory at high energies (see e.g. [32] and references therein).

Interestingly, one expects that the *absence* of single trace higher spin states in the low energy spectrum, namely the fact that all single trace higher spin operators have large anomalous dimensions, should impose an intriguing constraint: The theory has an AdS dual well approximated by Einstein gravity. For more discussion of this point see [33,34,26].

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Appendix A. Argument for $[Q_s, j_2] = \partial j_s + \cdots$

We now argue that the action of Q_s on the stress tensor produces j_s in the right hand side, where j_s is the current we used to construct Q_s ,

$$[Q_s, j_2] = \partial j_s + \dots, \qquad s > 1 \tag{A.1}$$

This follows from the fact that we know how the charge Q_s transforms under the conformal group.

The general form of the commutator is

$$[Q_s, T_{\mu\nu}] = c\partial_- J_{-\dots-\mu\nu} + \dots \tag{A.2}$$

Here we eliminated a possible term like $\partial_{\mu}J_{\nu-\dots}$ by applying ∂_{μ} to both sides of (A.2) and noticing that the left hand side, as well as the right hand side of (A.2) are zero, but the extra possible term would not be zero. Now, let us prove that c is not zero. Imagine it is zero, then contract (A.2) with dilaton Killing vector and integrate over $d^{d-1}x$ to get

$$[Q_{\zeta}, \int d^{d-1}x T_{0\nu} x^{\nu}] = [Q_{\zeta}, D] = 0$$
(A.3)

which is inconsistent with the fact that $[Q_{\zeta}, D] = (s-1)Q_{\zeta}$. Note that for s=1, the current does not appear in the right hand side. As usual the stress tensor should be invariant under global symmetries.

Appendix B. An integral expression for the odd three point functions

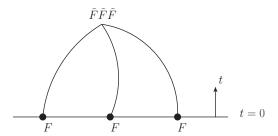


Fig. 1: Constructing the three dimensional correlator as arising from a four dimensional integral in a CFT. The three dimensional space lies at t = 0.

In this appendix we provide a "constructive" way to write it and a nice rationale for its existence. The idea is inspired by the AdS expressions for the odd pieces for spin one and spin two currents discussed in [35]. However, we will use something even simpler. We view the three point function as a correlator on a domain wall defect inside a (non-unitary) four dimensional conformal field theory. We construct it as follows. We start with four dimensional operators with spins $(j_l, j_r) = (s, 0)$, (0, s) at the four dimensional unitarity bound. We then introduce a quartic bulk interaction that preserves conformal symmetry. We then consider the correlator of three of these fields on a three dimensional subspace t = 0, to leading order in the interaction, see fig. 1.

We denote the four dimensional operators with spins $(j_l, j_r) = (s, 0)$, (0, s) by $F_{\alpha_1...\alpha_{2s}}$, $\tilde{F}_{\dot{\alpha}_1...\dot{\alpha}_{2s}}$. Of course, these operators are at the four dimensional unitarity bound and given in terms of free fields. The conformal dimension is [F] = s + 1 which is the same as dimension of conserved currents in d = 3 CFT. Their correlators are

$$\langle F_{\alpha_1...\alpha_{2s}}(x_1)\tilde{F}_{\dot{\alpha}_1...\dot{\alpha}_{2s}}(x_2)\rangle = \frac{\prod_{i=1}^{2s} (x_{12})_{\alpha_i\dot{\alpha}_i}}{(x_{12}^2)^{2s+1}}$$
(B.1)

we can contract indices with polarization spinors λ^{α} and $\bar{\mu}^{\dot{\alpha}}$ so that we get

$$\langle F(x_1, \lambda) \tilde{F}(x_2, \bar{\mu}) \rangle = \frac{(\lambda \not x_{12} \bar{\mu})^{2s}}{(x_{12}^2)^{2s+1}} = (\lambda \partial_{x_{12}} \bar{\mu})^{2s} \frac{1}{x_{12}^2}.$$
 (B.2)

Now let's consider the self dual interaction of the form

$$g \int dt d^3 \vec{z} \chi \tilde{F}_{s_1} \tilde{F}_{s_2} \tilde{F}_{s_3} \tag{B.3}$$

where all indices are contracted using $\epsilon^{\dot{\alpha}\dot{\beta}}$. Here χ is a scalar operator of dimension $[\chi] = 1 - (s_1 + s_2 + s_3)$. The contraction of indices is possible when the triangle inequality is satisfied

$$s_i \le s_{i+1} + s_{i+2}, \quad i = 1, 2, 3 \mod 3$$
 (B.4)

This interaction breaks parity in the bulk so that we can expect that it will generate parity breaking structure for the correlators at t = 0, where t is the fourth coordinate.

Now imagine we are computing correlation functions of three $F_{\alpha_1\cdots\alpha_{2s}}$ inserted at the boundary t=0. There is a subset of four dimensional conformal transformations which map this boundary to itself. They act on this boundary as the three dimensional conformal transformations. From that point of view the $F_{\alpha_1\cdots\alpha_{2s}}$ operators transform as three dimensional twist one operators of spin s. The contraction with the spinor λ_{α} gives us the same as contracting three dimensional currents with the three dimensional spinors, as in [27]. In addition, one can see that they are conserved currents from the three dimensional point of view²⁰. For this purpose it is enough to show that the propagator is annihilated by three dimensional divergence operator $\partial_{\lambda} \partial_{x} \partial_{\lambda}$. The structure of (B.2) shows that it is enough to check it for s=1. For s=1 the current is just $j_i=F_{ti}+\epsilon_{ijk}F_{jk}$. Then the divergence is zero due to the Maxwell equation and the Bianchi identity.

We now postulate that χ in (B.3) has expectation value $\langle \chi \rangle = t^{-\Delta_{\chi}} = t^{s_1+s_2+s_3-1}$. This expectation value is consistent with the three dimensional conformal invariance, and can be viewed as arising from a "domain wall defect" at t=0.

We can now consider the expression that results from considering the three point correlator at first order in the interaction (B.3)

$$\langle F_{s_1}(\vec{x}_1, \lambda_1) F_{s_2}(\vec{x}_2, \lambda_2) F_{s_3}(\vec{x}_3, \lambda_3) \rangle \sim \int_0^\infty dt d^3 \vec{x}_0 t^{s_1 + s_2 + s_3 - 1} \frac{(\lambda_1 \not x_{10} \not x_{02} \lambda_2)^{(s_1 + s_2 - s_3)} (\lambda_1 \not x_{10} \not x_{03} \lambda_3)^{(s_1 + s_3 - s_2)} (\lambda_2 \not x_{20} \not x_{03} \lambda_3)^{(s_2 + s_3 - s_1)}}{(t^2 + (\vec{x}_1 - \vec{x}_0)^2)^{2s_1 + 1} (t^2 + (\vec{x}_2 - \vec{x}_0)^2)^{2s_2 + 1} (t^2 + (\vec{x}_3 - \vec{x}_0)^2)^{2s_3 + 1}}$$
(B.5)

The operators in the left hand side are inserted at t = 0. They can be viewed as three dimensional conserved currents.

This is a conformal invariant expression for three dimensional correlators of conserved currents. It contains both even and odd contributions.

This needs to be checked separately because the interaction (B.3) is not obviously unitary and, thus, we cannot use the unitarity bound argument.

Notice that factors in the numerator of (B.5) have the schematic form

$$(\lambda_1 \not x_{i0} \not x_{0j} \lambda_2) \sim t(\vec{n} \vec{x}_{12}) + \epsilon_{ijk} n^i x_{01}^j x_{02}^k$$
(B.6)

where $\vec{n} = \lambda_1^{\alpha} \vec{\sigma}_{\alpha\beta} \lambda_2^{\beta} = \lambda_2^{\alpha} \vec{\sigma}_{\alpha\beta} \lambda_1^{\beta}$. We see that these two pieces behave differently under three dimensional parity. Thus, to extract the odd piece we can subtract parity conjugated integral, which can also be extracted by changing $t \to -t$. In the end one can show that the parity odd piece is given by extending the range of integration

$$\frac{\langle j_{s_1}(\vec{x}_1, \lambda_1) j_{s_2}(\vec{x}_2, \lambda_2) j_{s_3}(\vec{x}_3, \lambda_3) \rangle_{odd} \sim \int_{-\infty}^{\infty} dt d^3 \vec{x}_0 t^{s_1 + s_2 + s_3 - 1}}{(\lambda_1 \not x_{10} \not x_{02} \lambda_2)^{(s_1 + s_2 - s_3)} (\lambda_1 \not x_{10} \not x_{03} \lambda_3)^{(s_1 + s_3 - s_2)} (\lambda_2 \not x_{20} \not x_{03} \lambda_3)^{(s_2 + s_3 - s_1)}}{(x_{10}^2)^{2s_1 + 1} (x_{20}^2)^{2s_2 + 1} (x_{30}^2)^{2s_3 + 1}}.$$
(B.7)

This formula is valid both for integer and half integer spins. If $s_1 = s_2 = s$, then one can check that under the exchange $(\lambda_1, x_1) \leftrightarrow (\lambda_2, x_2)$ we get a factor of $(-1)^{s_3+2s}$. This implies that the correlator for two identical currents $\langle j_s j_s j_{s_3} \rangle$ is zero if s_3 is odd. Note that this is also zero when s is half integer, since in this case we expect an anticommuting result when we exchange the first two currents.

B.1. Light cone limit for the odd three point function

For the odd piece the light cone limit give us

$$\langle j_{s_1} j_{s_2} j_{s_3} \rangle = \partial_1^{s_1} \partial_2^{s_2} \Upsilon(s_3), \quad s_3 > 0$$
 (B.8)

where

$$\Upsilon(s) = \frac{1}{\hat{x}_{12}} \left[\frac{\hat{x}_{12}}{\hat{x}_{13} \hat{x}_{23}} \right]^s \sum_{k=0}^{\left[\frac{s-1}{2}\right]} {s-k-1 \choose k} \frac{1}{2k+1} \left[\frac{\hat{x}_{13} \hat{x}_{23}}{\hat{x}_{12}^2} \right]^k$$
(B.9)

here $[\alpha]$ means integer part of α and \hat{x} were defined in (5.3). And $\binom{n}{m}$ is the usual binomial coefficient. Interestingly, in this limit the answer can be written explicitly for any s. Notice that we do not have any square roots of the x_{ij} 's. Essentially for the same reason, we will not have a $|y_{12}|$ as we the limit $y_{12} \to 0$, we only get $1/y_{12}$. In general, it can be checked from the expressions in [27] that all square roots disappear from the odd correlator, even away from the light cone limit.

Appendix C. No odd piece from the one point energy correlator

Let's consider the case when $\langle j_2 j_2 j_2 \rangle$ is the boson piece plus the odd piece. In other words, $\langle j_2 j_2 j_2 \rangle_f = 0$.

Consider the one point energy correlator for the state created by a stress tensor at the origin with energy q^0 and zero spatial momentum. It is characterized by a two dimensional polarization tensor $\epsilon^{ij}T_{ij}$. See [4] for more details. The most general O(2) symmetric structure in this case is given by the following expression

$$\langle \mathcal{E}(\vec{n}) \rangle = \frac{q^0}{2\pi} \left[1 + t_4 \left(\frac{|\epsilon_{ij} n^j n^j|^2}{\epsilon_{ij} \epsilon_{ij}^*} - \frac{1}{4} \right) \right] - \frac{q^0}{2\pi} d_4 \left[\frac{\varepsilon^{ij} (n_i n^m \epsilon_{jm} \epsilon_{kp}^* n^k n^p + n_i n^m \epsilon_{jm}^* \epsilon_{kp} n^k n^p)}{2\epsilon_{ij} \epsilon_{ij}^*} \right], \tag{C.1}$$

it's even part was studied in [36], however, the odd piece was missed in the literature before as far as we know. The positive energy condition $\langle \mathcal{E}(\vec{n}) \rangle \geq 0$ becomes

$$|d_4| \le \sqrt{16 - t_4^2}. (C.2)$$

For the case of the boson it is easy to check that $t_4 = 4$. Thus, we are forced to set $d_4 = 0$. We got that in the theory of Majorana fermion $t_4 = -4$.

More intuitively consider the state created by $T_{11} - T_{22}$. Then the one point energy correlator is given by

$$\langle \mathcal{E}(\vec{n}) \rangle = \frac{q^0}{2\pi} \left[1 + \frac{t_4}{4} \cos 4\theta + \frac{d_4}{4} \sin 4\theta \right] \tag{C.3}$$

where θ is the angle between x^1 and the detector.

For a free boson, we have that the energy correlator vanishes at $\theta = \pi/4$ which forces $t_4 = 4^{-21}$. Then near $\theta = \pi/4$ we would have regions with negative energy correlator if d_4 was nonzero.

Thus, we are forced to conclude that the odd piece is not allowed by the assumption of the positivity of the energy flow. In the main text we have also excluded the odd piece in other ways.

²¹ The operator is odd under the exchange of the 1 and 2 axis, but the state of two massless scalars going back to back along this axis would be even.

Notice also that the energy correlator that we obtain, after setting the fermion and odd parts to zero, is such that it vanishes for one particular angle. This is already a suggestion that the theory is probably free, since in an interacting theory we expect showering so the energy distribution will never be exactly zero. In other words, showering from a neighboring angle would make it non-zero. In fact, it is natural to conjecture that if $\langle j_2 j_2 j_2 \rangle_f = 0$ (or $\langle j_2 j_2 j_2 \rangle_b = 0$), then the theory is free, without assuming that a higher spin current exist.

Appendix D. Half-integer higher spin currents implies even spin higher spin currents

Here we assume that we start from a higher, half-integer spin current j_s , $s \ge 5/2$. We then argue that we also have even spin currents. Then we can go back to the case treated in the main text.

The analysis is completely parallel to what was done in section five for integer spin currents. We simply need a couple of other formulas. For half integer s and s' we need that

$$\langle \underline{j_{s'}}\underline{j_{2}}_{b}\underline{j_{s}}\rangle \propto \partial_{1}^{s'-1/2}\partial_{2}^{2}\langle \underline{\psi}\phi\underline{j_{s}}\rangle_{free} ,$$

$$\langle \underline{j_{s'}}\underline{j_{2}}_{f}\underline{j_{s}}\rangle \propto \partial_{1}^{s'-1/2}\partial_{2}\langle \underline{\phi}\psi\underline{j_{s}}\rangle_{free} ,$$

$$\langle \underline{\psi}\phi\underline{j_{s}}\rangle_{free} \propto \frac{1}{\hat{x}_{13}^{3/2}\hat{x}_{23}^{1/2}} \left(\frac{1}{\hat{x}_{13}} - \frac{1}{\hat{x}_{23}}\right)^{s-1/2}$$

$$\langle \underline{\phi}\psi\underline{j_{s}}\rangle_{free} \propto \frac{1}{\hat{x}_{13}^{1/2}\hat{x}_{23}^{3/2}} \left(\frac{1}{\hat{x}_{13}} - \frac{1}{\hat{x}_{23}}\right)^{s-1/2}$$

$$\langle \underline{\phi}\psi\underline{j_{s}}\rangle_{free} \propto \frac{1}{\hat{x}_{13}^{1/2}\hat{x}_{23}^{3/2}} \left(\frac{1}{\hat{x}_{13}} - \frac{1}{\hat{x}_{23}}\right)^{s-1/2}$$
(D.1)

It is clear that in a free theory of bosons and fermions these formulas are true. In fact, these formulas follow from conformal invariance and current conservation of the third current. This can be shown directly using the methods of appendix I.

We can now use these expressions to consider the Q_s charge conservation identity on $\langle \underline{j_2j_2}_b j_s \rangle$. $[Q_s, j_s]$ gives a sum of integer spin currents so that we get a contribution to the charge conservation identity similar to the second term in (5.9). The action of $[Q_s, \underline{j_2j_2}_b]$ requires a bit more words. First $[Q_s, j_2]$ gives some other half integer spin currents. Then we can do the light-like OPE using (D.1) for each term and summing all the terms. After the dust settles, such terms give us

$$\langle [Q_s, \underline{j_2 j_2}_b] s \rangle = \partial_1^2 \partial_2^2 \left(\gamma \partial_1^{s - \frac{3}{2}} \frac{1}{\hat{x}_{13}} + \delta \partial_2^{s - \frac{3}{2}} \frac{1}{\hat{x}_{23}} \right) \frac{1}{\sqrt{\hat{x}_{12} \hat{x}_{13}}} \left(\frac{1}{\hat{x}_{13}} - \frac{1}{\hat{x}_{23}} \right)^{s - \frac{1}{2}}$$

$$\langle [Q_s, \underline{j_2 j_2}_f] s \rangle = \partial_1 \partial_2 \left(\gamma \partial_1^{s - \frac{1}{2}} \frac{1}{\hat{x}_{23}} + \delta \partial_2^{s - \frac{1}{2}} \frac{1}{\hat{x}_{13}} \right) \frac{1}{\sqrt{\hat{x}_{12} \hat{x}_{13}}} \left(\frac{1}{\hat{x}_{13}} - \frac{1}{\hat{x}_{23}} \right)^{s - \frac{1}{2}}$$
(D.2)

here permutation symmetry fixes $\delta = (-1)^{s-\frac{1}{2}}\gamma$.

The analysis of these charge conservation identities is very similar to the one we did before. Again we can count the number of independent terms in the right hand side and conclude that we expect a unique solution. This unique solution is the one we get in a free supersymmetric theory. We conclude that currents j_k appear in $\underline{j_2j_2}$ with even spins $k=2,4,\cdots,2s-1$. If $s\geq 5/2$, then $2s-1\geq 4$ and we have a higher spin current with integer spin and we can go back to the analysis done in section five.

Appendix E. Functions that are annihilated by all charges

Suppose that we have a function $f(x_i)$ of n variables that has a Taylor series expansion and is such that $Q_s = \sum_i (\partial_{x_i})^s$ annihilates it for all odd s. We want to prove that the function can be written as a sum of functions (not all equal)

$$f(x_i) = \sum_{\sigma} g_{\sigma}(x_{\sigma(1)} - x_{\sigma(2)}, x_{\sigma(3)} - x_{\sigma(4)}, ..., x_{\sigma(n-1)} - x_{\sigma(n)})$$
 (E.1)

where the sum is over all permutations. Notice that the each function depends on $\left[\frac{n}{2}\right]$ variables, compared to the original n x_i . If n is odd, then we drop the last variable (the function is independent of the last variable). Here we will not use conformal symmetry. Also we assume that the variables x_i are ordinary numbers and not vectors, etc.

E.1. Proof

First if f has a Taylor series expansion, we can organize it in terms of the overall degree of each term. Say we have a polynomial in x_i and we can separate the terms according to the overall degree of each term. Then Q_s should annihilate terms of different degrees. In other words, there is no mixing between terms of different degrees. Thus, we have to prove the statement for polynomials of degree k. The statement is obvious for degree k = 0. Now assume it is valid up to degree k and we want to prove it for degree k + 1.

If we have a polynomial P_{k+1} that is annihilated by the charges, then we can take $\partial_{x_1} P_k$ which is also annihilated and is a polynomial of degree k. Thus, by assumption $\partial_{x_1} P_k$ can be written as in (E.1). We can now integrate each term in such a way that we preserve its form. For example if we have a function of the form $g(x_1 - x_2, \cdots)$, then we integrate it with respect to x_{12} . After we are done, we would have written $P_{k+1} = \sum_{\sigma} g_{\sigma} + h(x_2, \cdots x_n)$. In other words, we are left with an integration "constant" which is a function of n-1 variables. Clearly the first term, the one involving the functions g, is annihilated by the Q_s , so h is also annihilated. Now we can repeat the argument for h, viewing it as a function of n-1 variables. In this way we eliminate all variables.

Thus, we have proven what we wanted to prove.

E.2. Functions with Fourier Transform

Of course, if the function f has a Fourier transform, then it is even easier to prove the statement. In that case we have $Q_s = \sum_i k_i^s$. Now we take $s \to \infty$, then the largest or lowest value of k dominates (depending on whether there is a |k| bigger than one or not). There must be an even number of largest $|k_i|$ otherwise for large s we would violate the equality. Say there are two, say $k_1 = -k_2$. Thus, the Fourier transform must have a $\delta(k_1 + k_2)$. We can now repeat the same with the second largest, etc, to argue that $f = \sum_{\sigma} g(k_{\sigma(2)}, k_{\sigma(4)}, \cdots, k_{\sigma(n)}) \delta(k_{\sigma(1)} + k_{\sigma(2)}) \cdots \delta(k_{\sigma(n-1)} - k_{\sigma(n)})$. which is the same as saying (E.1).

The reason we proved it for functions that have a Taylor series expansion, rather than a Fourier expansion, is that our definition of the quasi-bilocal operators, (5.14) and (5.15), involve the operator product expansion. So these are given in terms of power series expansions. In principle, we do not know how to continue them beyond their convergence radius. For this reason we do not know, in principle, whether a Fourier transform exists, or whether the action of the higher spin charges (5.16) continues to be valid beyond that region. Of course, the result proved in this appendix, together with conformal symmetry and analyticity, allows us to extend the bilocals everywhere.

Appendix F. Current conservation equation in terms of cross ratios

We consider parity even correlation functions of a conserved current with two spinning operators of general twist

$$\langle O_{\tau_1,s_1}(x_1)O_{\tau_2,s_2}(x_2)j_{s_3}(x_3)\rangle = \frac{1}{|x_{12}|^{\tau_1+\tau_2-1}|x_{13}|^{\tau_1+1-\tau_2}|x_{23}|^{\tau_2+1-\tau_1}}F(Q_i, P_i)$$
 (F.1)

where Q_i and P_i are the conformal cross ratios introduced in [27], see also [28]. Here $\tau_{2,3}$ are the twists of the second and third operators. The function F is constrained by current conservation. Here we give an expression for the condition for current conservation as a function of the cross ratios. The idea is to write $\partial_{\lambda_3} \partial_{x_3} \partial_{\lambda_3}$, act on (F.1), and then rewrite the answer in terms of cross ratios. We end up with an equation for F which can be expressed purely in terms of a differential operator acting on F, $\mathcal{D}_3 F = 0$, with

$$\begin{split} \mathcal{D}_{3} &= -8(Q_{2}Q_{3}-P_{1}^{2})\partial_{Q_{2}}\partial_{Q_{3}}^{2} + 8(Q_{1}Q_{3}-P_{2}^{2})\partial_{Q_{1}}\partial_{Q_{3}}^{2} \\ &-4Q_{2}\partial_{Q_{2}}\partial_{Q_{3}} + 4Q_{1}\partial_{Q_{1}}\partial_{Q_{3}} - Q_{1}\partial_{P_{2}}^{2} + Q_{2}\partial_{P_{1}}^{2} + \\ &+2(-Q_{1}P_{1}+P_{2}P_{3})\partial_{P_{1}}\partial_{P_{2}}^{2} - 2(-Q_{2}P_{2}+P_{1}P_{3})\partial_{P_{2}}\partial_{P_{1}}^{2} + \\ &+2(-Q_{1}Q_{2}+P_{3}^{2})(\partial_{P_{2}}^{2}\partial_{Q_{2}} - \partial_{P_{1}}^{2}\partial_{Q_{1}}) + \\ &+2(P_{2}^{2}-Q_{1}Q_{3})\partial_{Q_{3}}\partial_{P_{2}}^{2} - 2(P_{1}^{2}-Q_{2}Q_{3})\partial_{Q_{3}}\partial_{P_{1}}^{2} + \\ &+8(Q_{1}P_{1}-P_{2}P_{3})\partial_{Q_{1}}\partial_{Q_{3}}\partial_{P_{1}} - 8(Q_{2}P_{2}-P_{1}P_{3})\partial_{Q_{2}}\partial_{Q_{3}}\partial_{P_{2}} + \\ &+(\tau_{2}-\tau_{1})\left[-4\partial_{Q_{3}}Q_{3}\partial_{Q_{3}} - 4\partial_{Q_{3}}(P_{1}\partial_{P_{1}} + P_{2}\partial_{P_{2}}) - Q_{2}\partial_{P_{1}}^{2} - Q_{1}\partial_{P_{2}}^{2} - 2P_{3}\partial_{P_{1}}\partial_{P_{2}}\right] \end{split}$$
(F.2)

In principle one could also write a similar expression for odd structures. One can write the S_i in [27], by taking square roots in (2.18) of [27], and then being careful about the sign of the square root. Doing this, we can still use (F.2), for odd structures.

Appendix G. Four point function of scalar operators in the fermion-like theory

Using the transformation law for the twist two scalar which we derived in section 6.6

$$[Q_4, \tilde{j}_0] = \partial^3 \tilde{j}_0 + \gamma (\partial_-^2 j_{2-\perp} - \partial_- \partial_\perp j_{2--})$$
(G.1)

we can solve the charge conservation identity $\langle [Q_4, \tilde{j}_0 \tilde{j}_0 \tilde{j}_0 \tilde{j}_0 \tilde{j}_0] \rangle = 0$. The result is

$$\langle \tilde{j}_0 \tilde{j}_0 \tilde{j}_0 \tilde{j}_0 \tilde{j}_0 \rangle = \frac{f(u, v)}{x_{12}^4 x_{34}^4}$$

$$f(u, v) = f(v, u) = \frac{1}{v} f(\frac{u}{v}, \frac{1}{v})$$

$$f(u, v) = \alpha \left(1 + \frac{1}{u^2} + \frac{1}{v^2}\right) + \beta \frac{1 + u^{5/2} + v^{5/2} - u^{3/2} (1 + v) - v^{3/2} (1 + u) - u - v}{u^{3/2} v^{3/2}}$$
(G.2)

and for $\langle J_2 \tilde{j}_0 \tilde{j}_0 \tilde{j}_0 \rangle$ we get, using the notation of [28],

$$\langle J_{2}(x_{1})j_{0}(x_{2})j_{0}(x_{3})j_{0}(x_{4})\rangle = \frac{\epsilon(Z_{1}, P_{1}, P_{2}, P_{3}, P_{4})}{(P_{1}P_{2})^{3}(P_{1}P_{3})(P_{1}P_{4})(P_{3}P_{4})^{2}}$$

$$[Q_{123}f_{1}(u, v) + Q_{134}f_{2}(u, v) + Q_{142}f_{3}(u, v)],$$

$$Q_{1ij} = \frac{(Z_{1}P_{i})(P_{j}P_{1}) - (Z_{1}P_{j})(P_{i}P_{1})}{(P_{i}P_{j})},$$

$$\gamma f_{1}(u, v) = \frac{9}{5}\beta \frac{v^{3/2}}{u^{3/2}}, \ \gamma f_{2}(u, v) = \frac{9}{5}\beta \frac{u}{v^{3/2}}, \ \gamma f_{3}(u, v) = \frac{9}{5}\beta \frac{1}{u^{3/2}v^{3/2}}.$$
(G.3)

As in the case of bosons β and γ are fixed as soon as we choose the normalization for the two-point function of stress tensors.

Appendix H. Conformal blocks for free scalar

In the main text we introduced the quasi-bilocal operators by taking a particular light-like limit of stress tensors (5.14). These are the sum of contributions from several spins. They can be defined in any theory, even in theories that do not have the higher spin symmetry. In a generic interacting theory we only get the contribution of the stress tensor. Here we would like to present some explicit formulas showing that the projection on to a single spin contribution does not define a genuine bilocal operator.

Let us focus on the contribution with a particular spin s. This is again a quasi-bilocal operator which can be defined as follows

$$b_s(x_1, x_2) = \sum_{i \in S} c_{i,n} (x_1 - x_2)^i \partial^n j_s(\frac{x_1 + x_2}{2}),$$

$$\langle b_s(x_1, x_2) j_s(x_3) \rangle \sim \langle \phi^*(x_1) \phi(x_2) j_s(x_3) \rangle_{free}.$$
(H.1)

The second line fixes the constants in the first line. We can define it even away from the light cone limit by this formula. Defined in this way, $b_s(x_1, x_2)$ transforms as a bi-primary operator with conformal weights $(\Delta_x = \frac{1}{2}, \Delta_y = \frac{1}{2})$ and obeys the Laplace equation for both points. However, as we emphasized, this is not the product of two free fields!. In particular, notice that $b_2(x_1, x_2)$ exists in any CFT.

To see the quasi-bilocal nature of $b_s(x_1, x_2)$ let us consider the two point function of bilocals $\langle b_s(x_1, x_2)b_s(x_3, x_4)\rangle$. In d=3 it is given by the contribution of conserved current of spin s, J_s , into the four point function of free fields. This problem was addressed recently in [37]. The answer is given by the formula (6.20) in [37] with a=b=0, $\lambda_1=s+\frac{1}{2}$, $\lambda_2=\frac{1}{2}$, l=s

$$\langle b_s(x_1, x_2)b_s(x_3, x_4) \rangle = \frac{1}{|x_{12}||x_{34}|} \mathcal{F}_s(u, v)$$

$$\mathcal{F}_s(u, v) = \frac{\sqrt{u}}{\pi} \int_0^{\pi} d\theta \frac{4^s X^s (1 + \sqrt{1 - X})^{-2s}}{\sqrt{1 - X}}$$

$$X = z \cos^2 \theta + \bar{z} \sin^2 \theta$$

$$u = z\bar{z}, \quad v = (1 - z)(1 - \bar{z})$$
(H.2)

This integral is not known for general s. Fortunately, for s=0 it is known and we get

$$\langle b_0(x_1, x_2)b_0(x_3, x_4)\rangle = \frac{1}{|x_{13}||x_{24}|} \frac{2}{\pi} \frac{K(-\frac{z-\bar{z}}{1-z})}{\sqrt{1-z}}$$
(H.3)

where K(y) is the complete elliptic integral of the first kind.

One can check that this solution satisfies the Laplace equation. Also one can check that it has a singularity at the expected locations. However, the behavior near the singularity is not the one that is expected for the correlator of local operators. More precisely, in the limit $z \to 1$, with \bar{z} fixed but close to one, we get a term $\log(1-z)$. For local operators the singularities are power-like.

Appendix I. Fixing correlators in the light-like limit

Here we would like to show how the three point correlators are fixed in the light-like limit $\langle \underline{j_{s_1}j_{s_2}j_{s_3}}\rangle$. We use the conventions and notation of Giombi, Prakash and Yin [27]. In the light like limit we have $P_3 = 0$, the rest of the conformal invariants are non-zero.

The general conformal invariant, without any factor of P_3 has the form

$$\langle j_{s_1} j_{s_2} j_{s_3} \rangle = \frac{1}{|x_{12}||x_{13}||x_{23}|} \sum_{a,b} c_{ab} Q_1^a Q_2^b P_2^{2s_1 - 2a} P_1^{2s_2 - 2b} Q_3^{s_3 - s_1 - s_2 + a + b}. \tag{I.1}$$

We can now use the relation the P's and Q's given in equation (2.14) of [27]. Setting $P_3 = 0$ we get

$$P_1^2 Q_1 + P_2^2 Q_2 = Q_1 Q_2 Q_3 , \qquad \Rightarrow Q_3 = \frac{P_1^2}{Q_2} + \frac{P_2^2}{Q_1}$$
 (I.2)

We can use this relation to solve for Q_3 and eliminate it from (I.1). Then we find a simpler formula

$$\langle j_{s_1} j_{s_2} j_{s_3} \rangle = \frac{1}{|x_{12}||x_{13}||x_{23}|} f(s_1, s_2, s_3) ,$$

$$f(s_1, s_2, s_3) = \sum_{a=0}^{s_3} c(a) Q_1^{s_1 - a} P_2^{2a} P_1^{2(s_3 - a)} Q_2^{s_2 - s_3 + a}$$
(I.3)

We would like now to impose current conservation on the third current. The correspondent operator of the divergence can be obtained from (F.2) by setting P_3 to zero. The resulting operator is

$$\mathcal{D}_3 = -(1 + 2P_1\partial_{P_1} + 2Q_2\partial_{Q_2})Q_1\partial_{P_2}^2 + (1 + 2P_2\partial_{P_2} + 2Q_1\partial_{Q_1})Q_2\partial_{P_1}^2. \tag{I.4}$$

By imposing $\mathcal{D}_3 f(s_1, s_2, s_3) = 0$ we find a recursive relation

$$\frac{c(a+1)}{c(a)} = \frac{\left(s_1 + a + \frac{1}{2}\right)}{\left(a + \frac{1}{2}\right)} \frac{\left(s_3 - a - \frac{1}{2}\right)}{\left(s_3 + s_2 - a - \frac{1}{2}\right)} \frac{\left(s_3 - a\right)}{\left(a+1\right)} \tag{I.5}$$

This gives a unique solution, which is the one we considered in the main text (5.2). Once we know the solution is unique, we can simply use the free boson answer to get (5.2). But one can check explicitly that (I.5) gives (5.2).

I.1. Fermions

For fermions we expect a factor of P_3 . In whatever multiplies this factor, we can set $P_3 = 0$ and use (I.2). This means that the conservation condition will act in the same way on what multiplies P_3 as it acted on the boson case. Thus, we get the following structure

$$f^{f}(s_{1}, s_{2}, s_{3}) = P_{3}P_{1}P_{2}\sum_{b=0}^{\tilde{s}_{3}} \tilde{c}(b)Q_{1}^{\tilde{s}_{1}-b}P_{2}^{2b}P_{1}^{2(\tilde{s}_{3}-b)}Q_{2}^{\tilde{s}_{2}-\tilde{s}_{3}+b}$$
(I.6)

where $\tilde{s}_i = s_i - 1$. This looks very similar to (I.3). In fact, we see that

$$f^{f}(s_1, s_2, s_3) = P_3 f_h(s_1 - \frac{1}{2}, s_2 - \frac{1}{2}, s_3)$$
(I.7)

where f_h is the same sum as in (I.3) but with a running over half integer values $a = \frac{1}{2}, \frac{3}{2}, \dots, s_3 - \frac{1}{2}$. Thus, the action of \mathcal{D}_3 , (I.4), leads to the same recursion relation (I.5), but now running over half integer values of a.

This again proves that there is a unique structure, which is (5.4).

In the same way one can obtain (D.1).

I.2. General operators

We now explicitly solve the problem of finding the different parity even structures we can have for the three point function of two operators of the same twist and one conserved current. The problem is very similar to what we solved above. The main observation is that P_3 does not depend on λ_3 or x_3 . Then if we have a solution to the current conservation condition, we can generate a new solution by multiplying by P_3 . Then the general even structures have the form

$$\langle O_{s_1} O_{s_2} j_{s_3} \rangle \sim \frac{1}{|x_{12}|^{2\tau_0 - 1} |x_{23}| |x_{13}|} \sum_{l} P_3^{2l} \left[\langle j_{s_1 - l} j_{s_2 - l} j_{s_3} \rangle_b + \langle j_{s_1 - l} j_{s_2 - l} j_{s_3} \rangle_f \right]$$
(I.8)

The sum runs over the bosonic and fermionic three point functions of three conserved currents. We can easily check that these are solutions. The fact that these are all the solutions is obtained by taking light cone limits. To order P_3^0 the solution is constrained as in (I.5), which is the same as what we get from the l=0 in (I.8). If we now subtract the l=0 term, which is a solution, we can take the light cone limit and focus on the P_3^1 term, etc.

In (I.8), the sum over the bosonic structures runs over $1 + \min[s_1, s_2]$ values. The sum over fermion structures runs over $\min[s_1, s_2]$ values. The total number of structures that we have is

$$1 + 2\min[s_1, s_2] \tag{I.9}$$

These formulas are valid for integer spins. Probably, there is a similar story with half integer spins.

It would be interesting to fix also the structures that appear when the twists of the two operators are not the same. For that we would have to include the last term in (F.2) when we analyze the constraints.

Appendix J. More explicit solutions to the charge conservation identities of section five

Here we discuss further the charge conservation identities of section five and we prove that they have the properties stated there. To find the explicit solution of (5.9) it is convenient to restate the problem in a slightly different language. First remember that it is easy to prove that there is a unique solution of (5.9) which coincides with the free field one. However, we need to check that for all even $k \tilde{\alpha}_k$ is non-zero. So we focus on the free field theory.

J.1. Expression for the currents in the free theory

The expressions for the conserved currents are bilinear in the fields and contain a combination of derivatives that makes the field a primary conformal field. These are computed in multiple places in the literature, (see e.g. [38] [39] and references therein). We would like to start by providing a derivation for the expression for these currents that is particularly simple in three dimensions.

Consider the Fourier space expression for a current contracted with a three dimensional spinor λ , $\lambda^{2s}.J(q)$, where q^{μ} is the three-momentum. We can consider the matrix element

$$F = \langle p_1, p_2 | (\lambda^{2s}.J(q)) | 0 \rangle \tag{J.1}$$

here p_1 and p_2 are two massless three dimensional momenta of the two bosons or two fermions that make up the current. A massless three dimensional momentum can be written in terms of a three dimensional spinor, π_{α} . In other words, we write $p_{\alpha\beta} =$

 $p^{\mu}\sigma^{\mu}_{\alpha\beta} = \pi_{\alpha}\pi_{\beta}$. Since we have two massless momenta we have two such spinors π^1 and π^2 . Of course, $q = p_1 + p_2$ and it is not an independent variable. We can also view (J.1) as the form factor for the current. It is telling us how the current is made out of the fundamental fields. Clearly, Lorentz invariance implies that (J.1) is a function $F(\lambda.\pi^1, \lambda.\pi^2)$. However, since the current is also conserved, it should obey the equation

$$0 = q^{\alpha\beta} \frac{\partial^2}{\partial \lambda^{\alpha} \lambda^{\beta}} F = (\pi_1 \cdot \pi_2)^2 \left[\frac{\partial^2}{\partial (\pi^1 \cdot \lambda)^2} + \frac{\partial^2}{\partial (\pi^2 \cdot \lambda)^2} \right] F \tag{J.2}$$

where we expressed $q_{\alpha\beta} = \pi_{\alpha}^1 \pi_{\beta}^1 + \pi_{\alpha}^2 \pi_{\beta}^2$ and we acted on the function $F(\lambda.\pi^1, \lambda.\pi^2)$. We can now define the variables

$$z = \pi_1 \cdot \lambda - i\pi_2 \cdot \lambda$$
, $\bar{z} = \pi_2 \cdot \lambda + i\pi_2 \cdot \lambda$ (J.3)

Then the equation (J.2) becomes

$$\partial_z \partial_{\bar{z}} F(z, \bar{z}) = 0 \tag{J.4}$$

The solutions are purely holomorphic or purely anti-holomorphic functions. Since the homogeneity degree in λ is fixed to be 2s, we get the two solutions

$$F_b = z^{2s} + \bar{z}^{2s} , \qquad F_f = z^{2s} - \bar{z}^{2s}$$
 (J.5)

The first corresponds to the expression for the current in the free boson theory and the second to the expression in the free fermion theory, as we explain below. These expressions contain the same information as the functions $\langle \phi \phi^* j_s \rangle_{free}$ and $\langle \psi \psi^* j_s \rangle_{free}$ that we discussed in section five.

In order to translate to the usual expressions in terms of derivatives acting on fields, without loss of generality, we can choose λ so that it is purely along the direction λ^- . In the boson expression, each $\pi^i.\lambda$ appears to even powers, say $(\pi^1.\lambda)^{2k}(\pi^2.\lambda)^{2s-2k}$ such a term can be replaced by $\partial_-^k \phi^* \partial_-^{2(s-k)} \phi$. The F_b gives the expression

$$j_s^{bos} \sim \sum_{k=0}^{s} (-1)^k \frac{1}{(2k)!(2s-2k)!} \partial^k \phi^* \partial^{s-k} \phi.$$
 (J.6)

Similarly, for the fermion case we have odd powers and we replace $(\pi^1.\lambda)^{2k+1}(\pi^2.\lambda)^{2s-2k-1}$ by $\partial_-^k \psi_-^* \partial_-^{s-1-k} \psi_-$ and F_f in (J.5) gives

$$j_s^{fer} \sim \sum_{k=0}^{s} (-1)^k \frac{1}{(2k+1)!(2s-2k-1)!} \partial^k \psi^* \partial^{s-k} \psi.$$
 (J.7)

where all are minus component.

Notice that we do not even need to go to Fourier space to think about the z and \bar{z} variables. We can just think of them as book-keeping devices to summarize the expansions (J.7), (J.6) via the simple expressions (J.5).

J.2. Analyzing the charge conservation identity

Symmetries act on free field as

$$[Q_s, \phi] = \partial^{s-1}\phi, \quad [Q_s, \phi^*] = (-1)^s \partial^{s-1}\phi^*$$
 (J.8)

and a similar expression for the fermions. Expressing this in terms of the z and \bar{z} variables we rewrite (J.8) as

$$[Q_s, j_{s'}](z, \bar{z}) = [(z + \bar{z})^{2s-2} - (z - \bar{z})^{2s-2}] (z^{2s'} \pm \bar{z}^{2s'}).$$
 (J.9)

where the \pm indicate the results in a boson of fermion theory respectively. Then, using the notation from (5.9), we have

$$[Q_s, j_s](z, \bar{z}) = \sum_{k=1}^{s-1} \tilde{\alpha}_{2k} z^{2s-2k-1} \bar{z}^{2s-2k-1} (z^{4k} \pm \bar{z}^{4k})$$
 (J.10)

where we used the map $\partial_3 \to z\bar{z}$. As in the previous subsection this can be seen by $\partial_3 = \partial_{x_3^-} \to \lambda^2 q$ and $\lambda^2 \cdot q = (\lambda \cdot \pi^1)^2 + (\lambda \cdot \pi^2)^2 = z\bar{z}$. Expanding the left hand side of (J.9) for s' = s we get

$$\tilde{\alpha}_{2k} = \frac{2 \Gamma(2s-1)}{\Gamma(2k)\Gamma(2s-2k)} \tag{J.11}$$

which are all non-zero. Thus, this solves the problem of showing that $\tilde{\alpha}_k$ are non-zero for the free theories. This works equally simply for the boson-like and the fermion-like light like limits.

We can also do a similar expansion for the action of $[Q_s, j_{s'}]$ and find which terms appear in the right hand side

$$[Q_{s}, j_{s'}] = (2s - 2)! \sum_{r = -s' - s + 2}^{s + s' - 2} \tilde{\alpha}_{r} \partial^{s + s' - r - 1} j_{r}$$

$$\tilde{\alpha}_{r} = \frac{[1 + (-1)^{s + s' + r}]}{2} \left(\frac{1}{\Gamma(r + s - s')\Gamma(s + s' - r)} \pm \frac{1}{\Gamma(r + s + s')\Gamma(s - s' - r)} \right). \tag{J.12}$$

The \pm indicates the boson/fermion case.

This is how all the other charge conservation identities we discussed in section five can be analyzed. Namely, in section five we argued that all solutions are unique. And by computing for free theories as in this appendix we check which coefficients are non-zero. This works in the same way for the boson-like and fermion-like light like limits. We can go over the charge conservation identities and the properties we used.

- (5.19): We used that if s is odd, s > 1, and s' = 2, then both j_1 and j_s appear in the right hand side. This simply amounts to checking that for s' = 2, r = 1, s, the coefficient in (J.12) is non-zero for any odd s. We can also check that we get a j_s in the right hand side. This is the case both for the boson and fermion case.
- (5.21): We used that if s is even and s' = 2, then we get both a j_2 and a j_s in the right hand side of (J.12).
- (5.20): Here we used a slightly different property. The reason is that (5.20) does not correspond to the transformation of any free field, since an s even charge in free theories acts with + signs in front of the ∂^{s-1} . In order to analyze (5.20), it is convenient to view the functions in (5.20) in Fourier space, as we have done here. Then we can rewrite (5.20) as

$$\left[(z+\bar{z})^{2s-2} + (z-\bar{z})^{2s-2} \right] (z^2 \pm \bar{z}^2) = (2s-2)! \sum_{r=-s}^{s} \tilde{\beta}_r (z\bar{z})^{s-r} (z^{2r} \pm \bar{z}^{2r})$$
 (J.13)

Then one can find that

$$\tilde{\beta}_r = \frac{[1 + (-1)^{s+r}]}{2} \left(\frac{1}{\Gamma(r+s-1)\Gamma(s-r+1)} \pm \frac{1}{\Gamma(r+s+1)\Gamma(s-r-1)} \right)$$
(J.14)

which is the same expression as before, in (J.12), for s' = 1, except for the (-1) dependent prefactor which now implies that now the sum over r runs from $r = -s, -s+2, \dots, s$. Now the important point is that $\tilde{\beta}_s$ is non-zero. We used this in (5.20).

In the discussion of elimination of j'_0 , for the boson-like case, we used that $[Q_s, j_{s'}]$ produces both j_0 and j_2 in the boson theory and for s, s' even, with s > s'. This can also be checked from (J.12), by setting r = 0, 2. Note that in (J.12) $\tilde{\alpha}_0 = 0$ in fermion theory (the minus sign in (J.12)), as expected.

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